Integration by parts

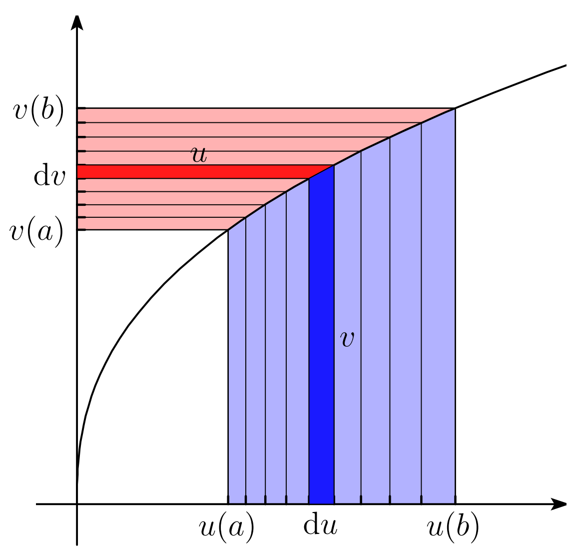
We give a geometric interpretation of the formula

$$\int_a^b u(x)v'(x)\,\mathrm{d}x + \int_a^b u'(x)v(x)\,\mathrm{d}x = u(b)v(b) - u(a)v(a), \tag{1}$$

which can also be written as

$$\int_{v(a)}^{v(b)} u\,\mathrm{d}v + \int_{u(a)}^{u(b)} v\,\mathrm{d}u = u(b)v(b) - u(a)v(a),$$

The illustration is for $u(x) = x^5$ and $v(x) = x^2$, but arbitrary differentiable functions could be used as well.



We consider the plane with axes u and v and draw the curve $x \mapsto (u(x), v(x))$ for values of x including the interval $[a, b]$.

For an (equidistant) grid of the interval $[a, b]$, we consider the grids that are induced by the functions $u(x)$ and $v(x)$ on the intervals $[u(a), u(b)]$ and $[v(a), v(b)]$, respectively. The first integral of (1) then corresponds to the red shaded region, and the second one to the blue shaded region. Their sum is seen to be the difference of the areas of two rectangles.

Genève, 1 August 2014 Ernst Hairer