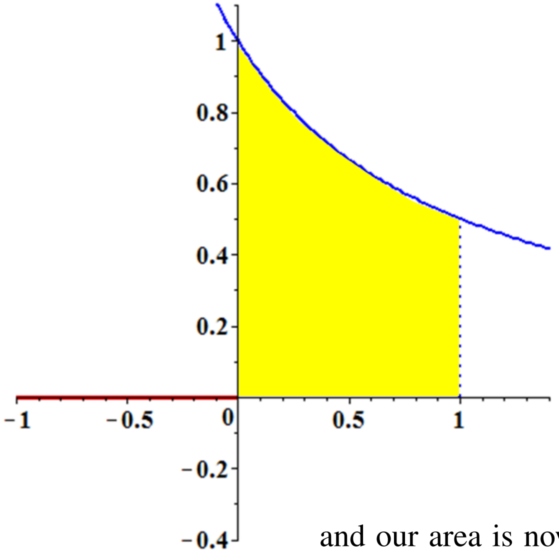
THEOREM OF THE DAY

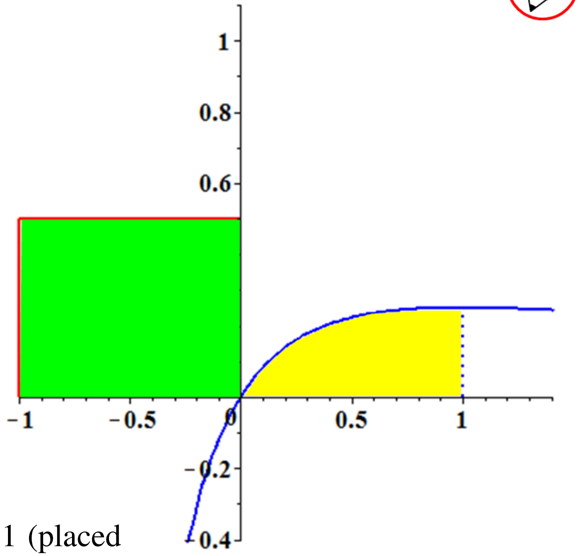
Integration By Parts *If $u(x)$ and $v(x)$ are differentiable functions of a real variable x, then*

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$



On the left the curve $y = \frac{1}{1+x}$ is plotted, with the shaded area being measured by the integral $\int_0^1 \frac{1}{1+x}\,dx$. Setting $u(x) = \frac{1}{1+x}$ and $v(x) = x$, we write

$$\int_0^1 \frac{1}{1+x}\,dx = \int_0^1 u\frac{dv}{dx}\,dx = [uv]_0^1 - \int_0^1 v\frac{du}{dx}\,dx = \left[\frac{x}{1+x}\right]_0^1 + \int_0^1 \frac{x}{(1+x)^2}\,dx,$$

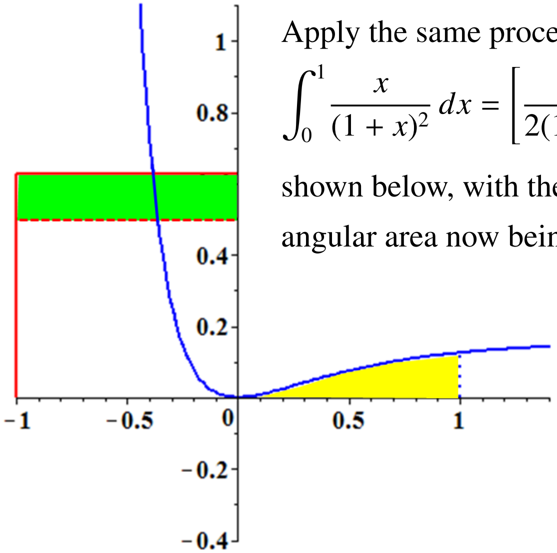


and our area is now measured as shown in the right-hand plot: $1/2$, shown as a rectangle of base 1 (placed for convenience to the left of the y-axis), plus the area between the curve $y = x/(1+x)^2$ and the x-axis, in the interval $[0, 1]$.

Apply the same process to the new integral:

$$\int_0^1 \frac{x}{(1+x)^2}\,dx = \left[\frac{x^2}{2(1+x)^2}\right]_0^1 + \int_0^1 \frac{x^2}{(1+x)^3}\,dx,$$

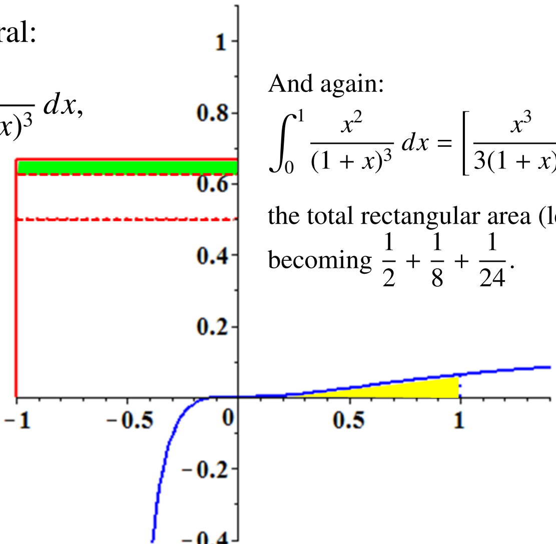
shown below, with the total rectangular area now being $\frac{1}{2} + \frac{1}{8}$.



And again:

$$\int_0^1 \frac{x^2}{(1+x)^3}\,dx = \left[\frac{x^3}{3(1+x)^3}\right]_0^1 + \int_0^1 \frac{x^3}{(1+x)^4}\,dx,$$

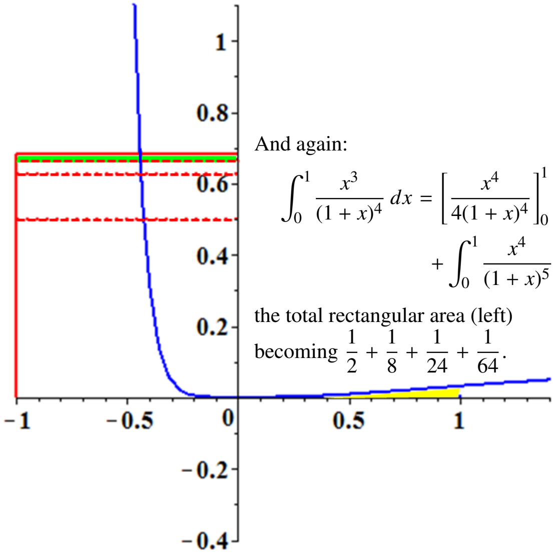
the total rectangular area (left) becoming $\frac{1}{2} + \frac{1}{8} + \frac{1}{24}$.



And again:

$$\int_0^1 \frac{x^3}{(1+x)^4}\,dx = \left[\frac{x^4}{4(1+x)^4}\right]_0^1 + \int_0^1 \frac{x^4}{(1+x)^5}\,dx,$$

the total rectangular area (left) becoming $\frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64}$.



More generally we can, in the limit, write $\ln(1+t) = \int_0^t \frac{1}{1+x}\,dx = \frac{t}{1+t} + \frac{t^2}{2(1+t)^2} + \frac{t^3}{3(1+t)^3} + \ldots$, and this converges provided $|t| < |t+1|$.

Integration by parts is often taught as a ‘trick’ for extracting antiderivatives of stubborn functions: $e^x \sin x$, $\sqrt{1+4x^2}$, etc. But its origins, which are generally traced back to Brook Taylor in 1715, lie in evaluating infinite summations.

Web link: Brook Taylor’s *Methodus Incrementorum*, translated and annotated by Ian Bruce: www.17centurymaths.com; Prop. XI. Theor. IV.
Further reading: *Analysis by its History* by Ernst Hairer and Gerhard Wanner, Springer, 1996.