THEOREM OF THE DAY

The Contraction Mapping Theorem *Let M be a nonempty complete metric space and let f be a contraction mapping of M to itself. Then f has precisely one fixed point, i.e. there is a unique point $x \in M$ such that $f(x) = x$.*

If M's a complete metric space,
And non-empty, we know it's the case
That if f's a contraction
Then under its action
Just one point remains in its place.

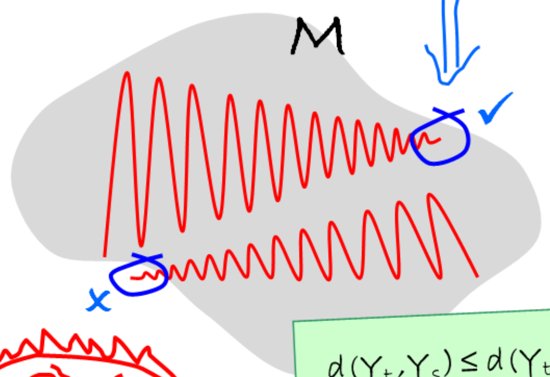
Metric space has distance function $d(x,y)$ satisfying triangle inequality $d(x,y)+d(y,z) \geq d(x,z)$

$d(P,Q) = d(f(P),f(Q)) \leq C.d(P,Q) < d(P,Q)$

Contraction $= \exists\, C$ in $[0,1)$ s.t. for any 2 points x and y, $d(f(x),f(y)) \leq C.d(x,y)$

Complete = all limits of Cauchy sequences in M are also in M

$Y_0 \xrightarrow{f} Y_1 \xrightarrow{f} Y_2$



$\sum_{b=t+1}^{s} d(Y_{b-1},Y_b) \geq d(Y_t,Y_s)$

$\underset{k}{d(Y_0,Y_1)} \xrightarrow{f} \underset{\leq Ck}{d(Y_1,Y_2)} \xrightarrow{f} \underset{\leq C^2k}{d(Y_2,Y_3)} \ldots$

$d(Y_t,Y_s) \leq d(Y_t,Y_{t+1}) + d(Y_{t+1},Y_{t+2}) + \ldots$
$\leq C^tk + C^{t+1}k + C^{t+2}k + \ldots = kC^t/(1-C)$

$f(P) = \lim_{n\to\infty} f(Y_n) = \lim_{n\to\infty} Y_{n+1} = P$

Stefan

PROOF

First suppose to the contrary two points don't move:
$f(Q)$ equals Q, $f(P)$ equals P.
Then consider the distance PQ: we can prove
That this distance is less that itself, which can't be.

Thus uniqueness; existence takes longer to get:
We'll construct such a point, fixed by f, as is sought.
Let us first take a point, say Y_0, in the set
And let Y_n be $f^n(Y_0)$.

By the triangle law, given t less than s,
Summing lengths from Y_{b-1} to Y_b
Over b more than t up to s, gives not less
Than the length to Y_s all the way from Y_t.

Now suppose that f's constant is capital C,
And the distance Y_0 to Y_1 is called k;
Then this sum is not more than the sum, a from t
Up to $(s-1)$, of kC^a.

This is less than or equal to k times the sum
Of all C^a for which a's at least t
And this last sum, by easy summation, will come
To $kC^t/(1-C)$.

So the sum tends to zero, and (Y_n) is then
Clearly Cauchy, and so it converges. Now see
That by f's continuity, $f(Y_n)$
Tends to f of the limit of Y's—call it P.

But now $f(Y_n)$ equals Y_{n+1},
So this P's fixed by f as required, so we're done.

The contraction mapping theorem is due to the Polish mathematician Stefan Banach in 1922 and has many applications, notably to proving the existence and uniqueness of solutions to differential equations. The clever limerick statement by Dilip Sequeira and the splendid proof rejoinder of Michael Fryers are reproduced with permission from *Eureka*, the magazine of the Archimedeans, no. 52, 1993, p. 3, and no. 53, 1994, p. 53, respectively.

Web link: www.tricki.org/article/How_to_use_fixed_point_theorems

Further reading: *Introduction to Metric and Topological Spaces, 2nd ed.* by Wilson A. Sutherland, Oxford University Press, 2009.