# THEOREM OF THE DAY

L'Hospital's Rule *Let $f(x)$ and $g(x)$ be real-valued functions of a real variable x. Let $a \in \mathbb{R}$ and suppose that, in some open interval containing a, $f(x)$ and $g(x)$ are differentiable except possibly at $x = a$, with $g'(x)$ being nowhere zero. Suppose, moreover that $\lim\limits_{x \to a} f = \lim\limits_{x \to a} g = 0$ or $|\lim\limits_{x \to a} f| = |\lim\limits_{x \to a} g| = \infty$. Then $\lim_{x \to a} f'(x)/g'(x)$, provided this limit exists, satisfies*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

* L'Hospital and circularity
Differentiating 'from first principles', to obtain, e.g., sin′ = cos, involves taking limits. The consequent danger of circularity in applying L'Hospital (see, e.g., Question 117124 at math.stackexchange.com) may be circumvented using geometry. See, e.g., Tim Hesterberg's The Spider's Spacewalk.

Legend:
- $\cos x$ (yellow)
- $\dfrac{\sin x}{x}$ (red)
- $x \ln x$ (blue)
- x^x (green)

$\tau = 2\pi$
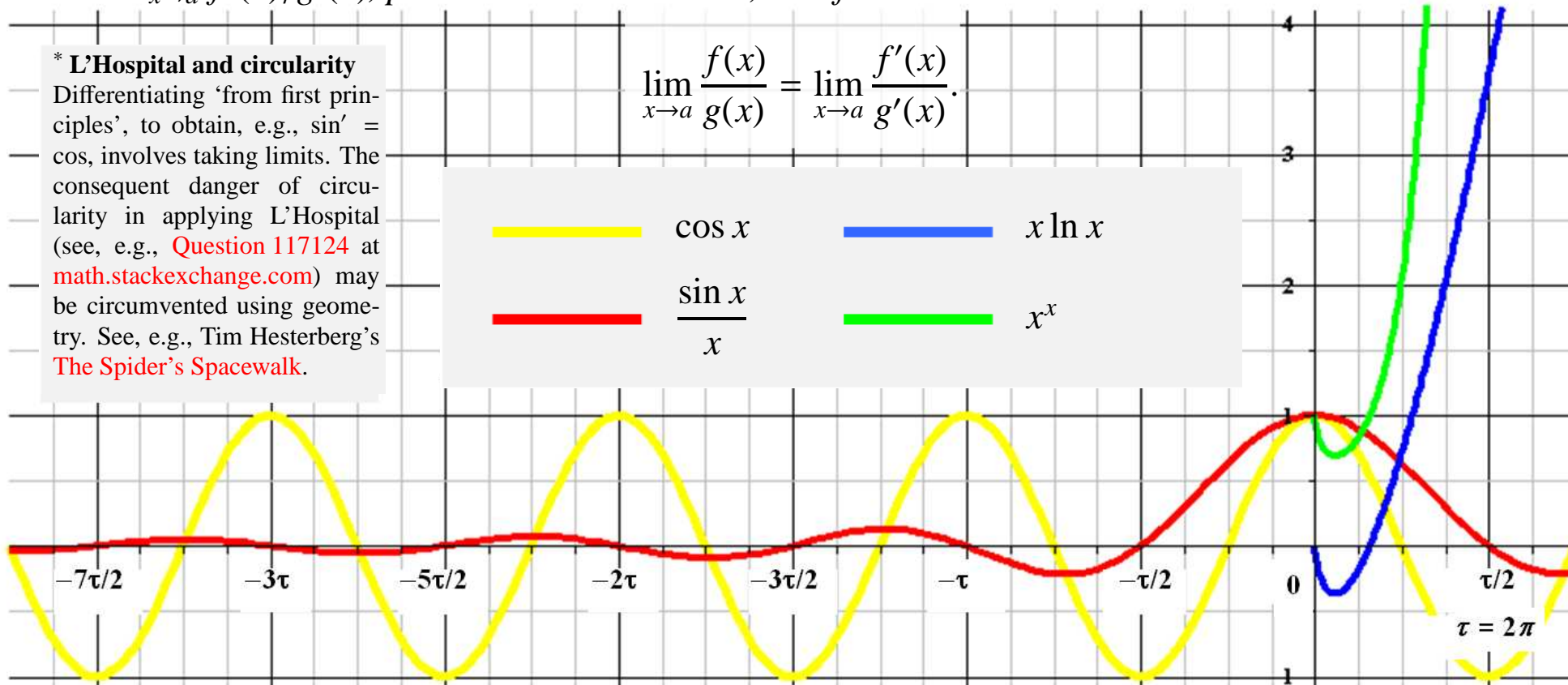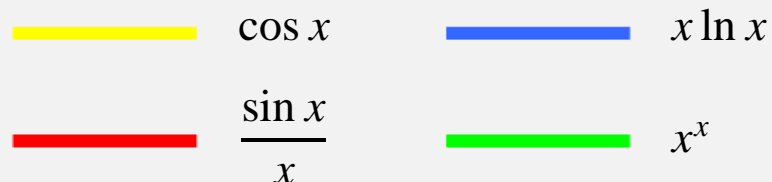
It is easy to see that as x approaches 0, the value of $\cos x$ approaches 1, i.e. $\lim_{x \to 0} \cos x = 1$. Similarly $\lim_{x \to 0} \sin x = 0$ and $\lim_{x \to 0} x = 0$, but what about the ratio of these two functions? The '0/0' version of L'Hospital's Rule applies* and gives this limit as $\lim_{x \to 0} (d \sin x/dx)/(d\, x/dx) = \lim_{x \to 0} \cos x/1 = 1$.

L'Hospital's Rule works equally well with one-sided limits. For example, $x \ln x$ is only defined as we approach zero from the positive direction. We tackle $\lim_{x \to 0^+} x \ln x$ with a trick: rewrite it as $\lim_{x \to 0^+} \ln x/(1/x)$ Now $\lim_{x \to 0^+} \ln x = -\infty$ and $\lim_{x \to 0^+} 1/x = \infty$, the '$\infty/\infty$' version of L'Hospital's Rule applies and gives $\lim_{x \to 0^+} (d \ln x/dx)/(d (1/x)/dx) = \lim_{x \to 0^+} (1/x)\big/(-1/x^2) = \lim_{x \to 0^+} -x = 0$.

We have a nice corollary using the fact that $\exp(\lim) = \lim(\exp)$: thus $0^0 = \lim_{x \to 0^+} x^x = \lim_{x \to 0^+} \exp(\ln(x^x)) = \exp(\lim_{x \to 0^+} \ln(x^x)) = \exp(\lim_{x \to 0^+} x \ln x) = e^0 = 1$.

A corollary of the Mean Value Theorem, given by history to Johann Bernoulli (c. 1690); but by posterity to Guillaume de L'Hospital.

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