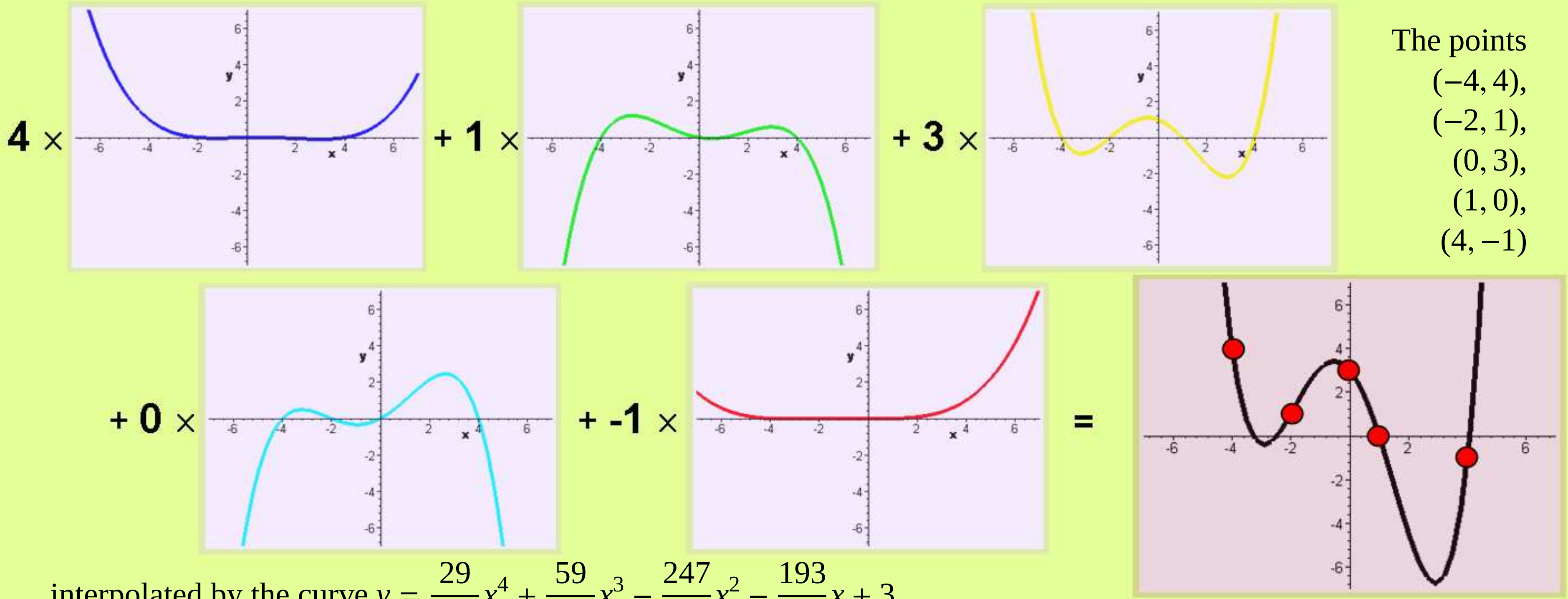
THEOREM OF THE DAY

The Lagrange Interpolation Formula *Given n distinct real values, $x_1, \ldots, x_n$, $n \geq 2$, and any n points, $(x_1, y_1), \ldots, (x_n, y_n)$, in the Cartesian plane, there is unique polynomial curve, $y = p(x)$, of degree $n - 1$, passing these points, specified by*

$$p(x) = \sum_{i=1}^{n} y_i \prod_{i \neq j} (x - x_j)/(x_i - x_j).$$



... interpolated by the curve $y = \frac{29}{320}x^4 + \frac{59}{960}x^3 - \frac{247}{160}x^2 - \frac{193}{120}x + 3$.

Remarks:

1. Writing the formula explicitly for $n = 2$ points gives $y = y_1(x - x_2)(x_1 - x_2)^{-1} + y_2(x - x_1)(x_2 - x_1)^{-1}$, the equation for the unique straight line passing through these points: with a little manipulation, it becomes the more memorably symmetrical straight line equation $(y - y_1)/(y_2 - y_1) = (x - x_1)/(x_2 - x_1)$.
2. The $n = 2$ calculation looks very similar to that which solves the Chinese Remainder Theorem for two modular equations, and indeed there is a close connection.
3. The calculation also reveals why the formula works in general: each term is a polynomial which takes the value y_i when $x = x_i$ and is zero when $x = x_j$, $j \neq i$.
4. We can deduce uniqueness thus: suppose $p(x)$ and $q(x)$ are polynomials through our n points, and define the polynomial $r(x) = p(x) - q(x)$. Now for $i = 1, \ldots, n$, $r(x_i) = 0$ so, by the Factor Theorem, $(x - x_i)$ is a factor of $r(x)$. So $r(x)$ looks like $(x - x_1)(x - x_2) \cdots (x - x_n) \times s(x)$, for some polynomial $s(x)$. Expanding the brackets gives $r(x)$ a term $x^n s(x)$ which has higher degree of p and q. This is impossible so $s(x)$, and hence $r(x)$, must be the zero polynomial.

Edward Waring extracted this formula in 1779 from a more general one of Newton. Independently, Lagrange did likewise in 1795.

Web link: imagescience.org/meijering/publications/1015/

Further reading: *Over and Over Again* by Gengzhe Chang and Thomas W. Sederberg, MAA, 1998, chapter 17.