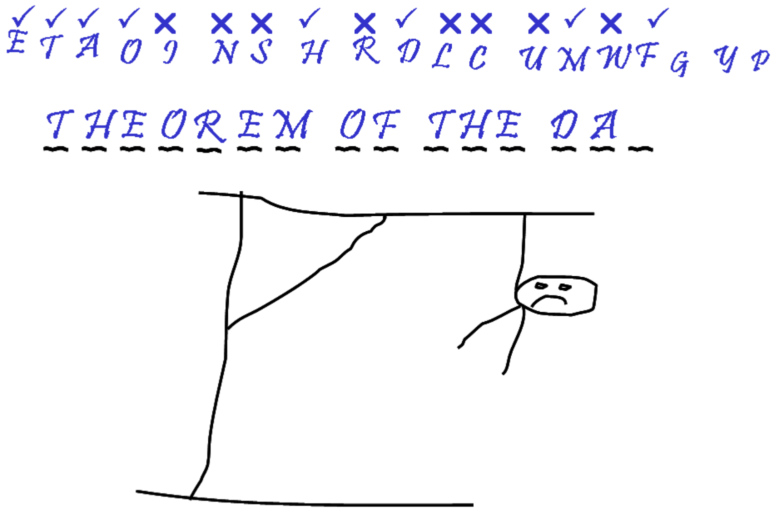
THEOREM OF THE DAY

Zeckendorf's Theorem *Every positive integer may be represented in a unique way as a sum of distinct non-consecutive Fibonacci numbers. More precisely, given the sequence of distinct Fibonacci numbers:* $F_1 = 1, F_2 = 2$ *and, for* $k \geq 3$, $F_k = F_{k-1} + F_{k-2}$, *and given any positive integer* N, *there is a unique, finite binary string* $b_1 b_2 \ldots b_t, t \geq 1$, *having no consecutive ones, such that* $N = \sum_{i=1}^{t} b_i F_i$.

How to win at Hangman…



…and do data compression

ORDER	LETTER	CODE	ORDER	LETTER	CODE
1	E	11	14	M	1000011
2	T	011	15	W	0100011
3	A	0011	16	F	0010011
4	O	1011	17	G	1010011
5	I	00011	18	Y	0001011
6	N	10011	19	P	1001011
7	S	01011	20	B	0101011
8	H	000011	21	V	00000011
9	R	100011	22	K	10000011
10	D	010011	23	J	01000011
11	L	001011	24	X	00100011
12	C	101011	25	Q	10100011
13	U	0000011	26	Z	00010011

Letters, in everyday use, occur with very pronounced frequencies: E most commonly, then T, then A, etc. You should always try the letters in this order when playing the game of Hangman — at least until you can start guessing the answer. And sending digital messages (using binary digits) is quicker if the letters are encoded with fewer bits for the more frequent letters. In the theorem, the number 17 gives binary string $b_1 = 1, b_2 = 0, b_3 = 1, b_4 = 0, b_5 = 0, b_6 = 1$ since $1 \times F_1 + 0 \times F_2 + 1 \times F_3 + 0 \times F_4 + 0 \times F_5 + 1 \times F_6 = 1 + 3 + 13 = 17$. In *Fibonacci coding*, invented by Alberto Apostolico and Aviezri Fraenkel in 1985, the string 101001 can encode G, the 17th most common letter, an extra 1 being added at the end to signal letter boundaries. 'THEOREM' is encoded in standard ASCII as seven 8-bit bytes: 56 bits. The Fibonacci code makes a 46% saving with only 30 bits: 011000011111011100011111000011.

In 1960, David E. Daykin proved that the Fibonacci sequence is the *only* one which satisfies Zeckendorf's theorem: *if another sequence* $(G_n)_{n\geq 1}$ *gives unique representation by sums of non-consecutive terms then* $G_n = F_n$ *for all n.* Édouard Zeckendorf, a Belgian amateur mathematician, discovered his theorem in 1939, but it appears to have first been published in 1952 by Cornelis Lekkerkerker, who derived the remarkable average: *if* $s(n)$ *denotes the number of terms in the Zeckendorf representation of n, and* $S(n) = \frac{1}{F_n}\sum_{k=F_{n+1}}^{F_{n+2}-1} s(k)$ *then* $\lim_{n\to\infty} S(n)/n = 1/(1+\varphi^2)$, $\varphi = \left(1+\sqrt{5}\right)/2$ *(the golden ratio).*

Web link: www.ics.uci.edu/~dan/pubs/DC-Sec3.html. And check out Colm Mulcahy's wonderful Zeckendorf-based card magic: additional-certainties at www.maa.org/community/maa-columns/past-columns-card-colm.

Further reading: *The Golden Ratio and Fibonacci Numbers* by R.A Dunlap, World Scientific, 1998.