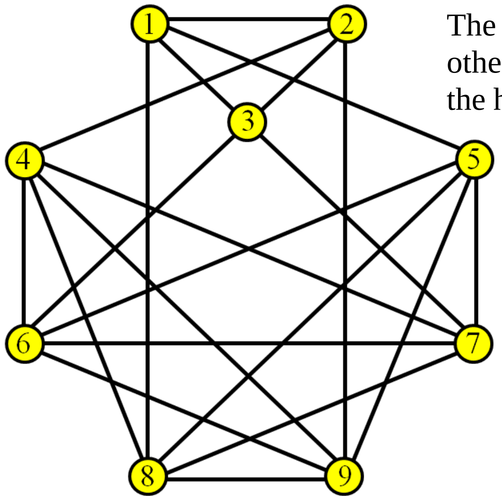
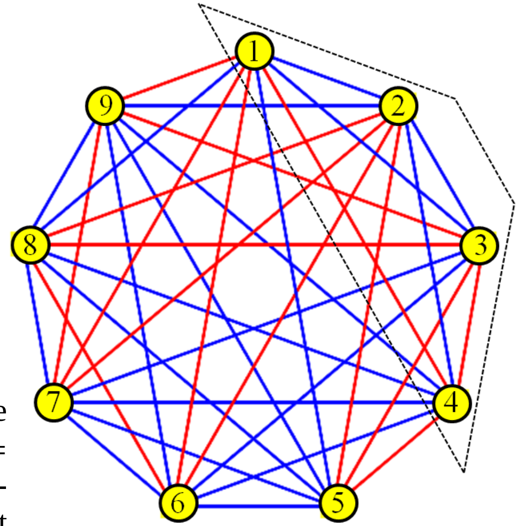
THEOREM OF THE DAY

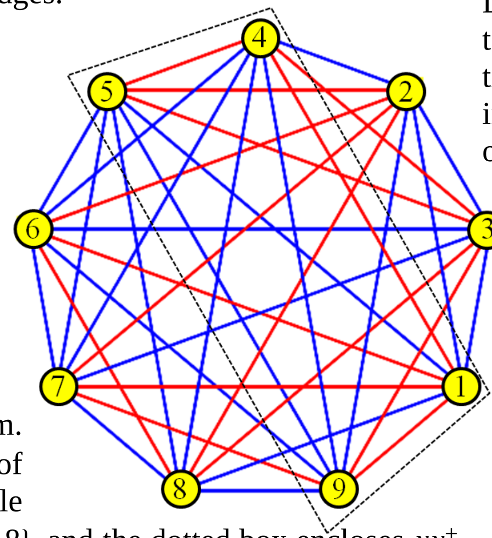
Ore's Theorem in Graph Theory *Let G be a simple connected graph with n vertices, $3 \le n$. Suppose that for any pair of nonadjacent vertices their degrees total at least n. Then G has a Hamilton cycle.*



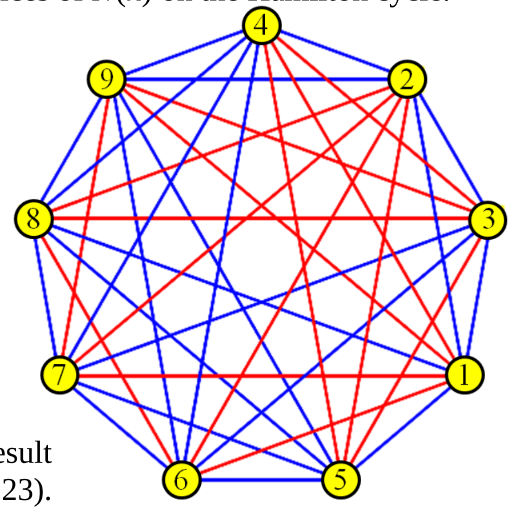
The graph shown on the left has $n = 9$ vertices, with vertices 1, 2 and 3 having degree (number of incident edges) equal to 4 while all other vertices have degree 5. Any two nonadjacent vertices have degree sum 9 (e.g. vertices 3 and 4) or 10 (e.g. vertices 4 and 5). So the hypothesis of the theorem holds and a Hamilton cycle (closed sequence of edges visiting each vertex exactly once) exists.



A neat proof, due to J.A. Bondy, translates into an algorithm for finding the Hamilton cycle: firstly take the complete graph K_n on n vertices and represent G as a subgraph by colouring an appropriate subset of edges blue, as shown on the left. Now K_n certainly has a Hamilton cycle: in our drawing we may take the 'outside' cycle. If all its edges are blue then we are done. But in our case this fails for three edges.



Let xx^+ be the first red edge on our Hamilton cycle. Let $N(x)$ and $N(x^+)$ be, respectively, the sets of neighbours of x and x^+ in G, and let $N^+(x)$ be the set of successors of vertices of $N(x)$ on the Hamilton cycle.



In our example (immediate right) we have $xx^+ = 34$, with $N(3) = \{1, 2, 6, 7\}, N(4) = \{2, 6, 7, 8, 9\}$ and $N^+(3) = \{2, 3, 7, 8\}$. Observe that $N(x^+) \cap N^+(x) \ne \emptyset$. This must be the case! For suppose that $N(x^+)$ includes no vertex of $N^+(x)$. Then

$$|N(x^+)| \le n - 1 - |N^+(x)| = n - 1 - |N(x)|.$$

But this would mean that $|N(x)| + |N(x^+)| < n$, contradicting the hypothesis of the theorem. Suppose that $y \in N(x)$ has successor $y^+ \in N(x^+) \cap N^+(x)$. Then xx^+ and yy^+ are edges of the Hamilton cycle at least one of which is red, while xy and x^+y^+ are chords of the cycle which are both blue. In the graph above, we may take $y^+ = 2 \in N(4) \cap N^+(3) = \{2, 7, 8\}$, and the dotted box encloses $xx^+ = 34$, $yy^+ = 12$, $xy = 31$ and $x^+y^+ = 42$.

And now we may increase the number of blue (G) edges in our Hamilton cycle: discard xx^+ and yy^+ in favour of xy and x^+y^+. The result for our example is shown above right (the new Hamilton cycle has been kept as the outside cycle by 'flipping' the K_n about the edge 23).

A new red cycle edge, 19, is now chosen. The same procedure as before pairs it with edge 54 which luckily happens to be the last remaining red cycle edge. So the resulting replacement discards two red cycle edges and produces the blue Hamilton cycle (far right).

Oystein Ore published this classic result in a 1-page note in 1960, generalising a 1952 theorem of Gabriel Dirac: *hamiltonicity of an n-vertex graph is implied by a minimum degree of not less than $n/2$* (note that this stronger hypothesis fails for our example graph). An influential and surprising 1971 theorem of Adrian Bondy in turns strengthens Ore's conclusion: *except in the case of complete regular bipartite graphs, not only hamiltonicity but pancyclicity (cycles of every length ≥ 3) follows*! You may confirm this for our example.

Web link: scholar.rose-hulman.edu/rhumj/vol1/iss1/6/

Further reading: *Graphs and Digaphs* by Gary Chartrand, Linda Lesniak and Ping Zhang, Chapman and Hall/CRC, 5th Edition, 2010.