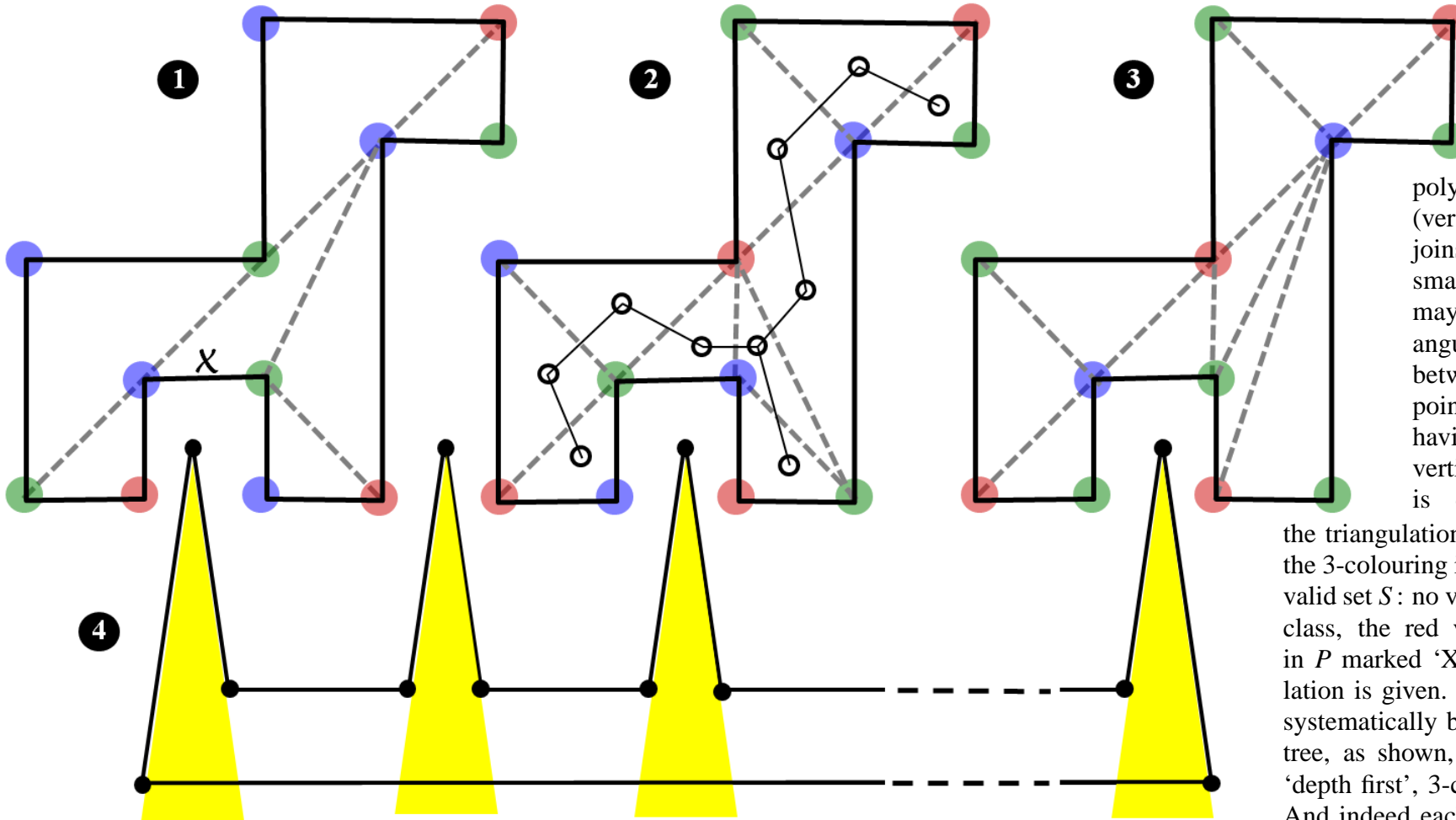




THEOREM OF THE DAY

The Art-Gallery Theorem *Let P be the subset of the Euclidean plane consisting of an n -vertex simple polygon and its interior. Then P contains a finite subset S , of cardinality at most $\lfloor n/3 \rfloor$, such that every point of P is joined to some point of S by a straight line contained in P .*



The examples on the left are based on Steve Fisk's beautiful 1978 proof of this theorem.

If a triangulation of the polygon is properly 3-coloured (vertices coloured so that no edge joins the same colours) then any smallest colour class of vertices may form our set S . Note that 'triangulation' means adding edges between vertices so that every point of P belongs to a triangle having precisely three polygon vertices. Thus, at ❶, although P is divided up into triangles,

the triangulation is incomplete; and although the 3-colouring is valid it does not guarantee a valid set S : no vertex from the smallest colour class, the red vertices, can 'see' the point in P marked 'X'. At ❷, a complete triangulation is given. The 3-colouring is produced systematically by joining the triangles into a tree, as shown, and then traversing the tree 'depth first', 3-colouring triangle by triangle. And indeed each colour class is a valid can-

didate for set S and has cardinality $\lfloor 12/3 \rfloor = 4$. However, not all triangulations are equal! The one at ❸ produces a set S (the blue vertices) which is optimal, having cardinality 2. So we can sometimes do better than $\lfloor n/3 \rfloor$; but not always—in the example at ❹ each triangle necessarily adds an extra point to set S .

This theorem was published by Václav Chvátal in 1975 in response to a question by Victor Klee. The lower bound can, in the words of Chvátal's original paper "be interpreted as the minimum number of guards required to supervise any art gallery with n walls."

Web link: www.ams.org/samplings/feature-column/fcarc-diagonals1. See www.ams.org/samplings/feature-column/fcarc-klee for historical context.

Further reading: *Discrete and Computational Geometry*, by Satyan L. Devadoss and Joseph O'Rourke, Princeton University Press, 2011, chapter 1.

