A classic example is the measurement of the surface area and volume of a torus. A torus may be specified in terms of its minor radius $r$ and major radius $R$ by rotating through one complete revolution (an angle of $\tau$ radians) a circle of radius $r$ about an axis lying in the plane of the circle and at perpendicular distance $R$ from its centre. In the image on the right, the surface and volume being generated by the rotation have area $A = \tau r \times \tau R = \tau^2 R$ and volume $V = \frac{1}{2} \tau r^2 \times \tau R = \frac{1}{2} \tau^2 r^2 R$, respectively.

Using calculus, the centroid of the region bounded by the curve $y = f(x)$ and the $x$-axis in the interval $[a, b]$ has $x$ and $y$ coordinates

$$
\bar{x} = \frac{1}{A} \int_a^b x f(x) \, dx \quad \text{and} \quad \bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} (f(x))^2 \, dx,
$$

where $A$ is the area of the region. Now the second Pappus–Guldin theorem gives the volume when this region is rotated through $\tau$ radians as $V = A \times \tau \bar{y} = \frac{1}{2} \tau \int_a^b (f(x))^2 \, dx$, the familiar formula for volume of solid of revolution. A similar calculation may be made using the $y$ coordinate of the centroid of the arc on the curve $y = f(x)$, on the interval $[a, b]$, the coordinates of this centroid being given as:

$$
\bar{x} = \frac{1}{L} \int_a^b x \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \quad \text{and} \quad \bar{y} = \frac{1}{L} \int_a^b y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx,
$$

with $L$, the arc length, calculated as $\int_a^b \sqrt{1 + (dy/dx)^2} \, dx$. This is illustrated, right, for a semicircle: $y = \sqrt{1 - x^2}$, in the interval $[-1, 1]$. The centroid of the enclosed region is $(0, 8/3\tau)$, plotted as the outline circle; the centroid of the semicircular arc is $(0, 4/\tau)$, plotted as the solid circle.

Pappus stated his theorems in the early 300s; it is accepted that 17th century scientists rediscovered the theorems for themselves, Book II (1640) of *Centrobarica*, Paul Guldin’s 700-page treatise on centres of gravity, being the pre-eminent contribution.


**Further reading:** *Philosophy of Mathematics & Mathematical Practice in the Seventeenth Century* by Paolo Mancosu, Oxford University Press, 1999, chapter 2.