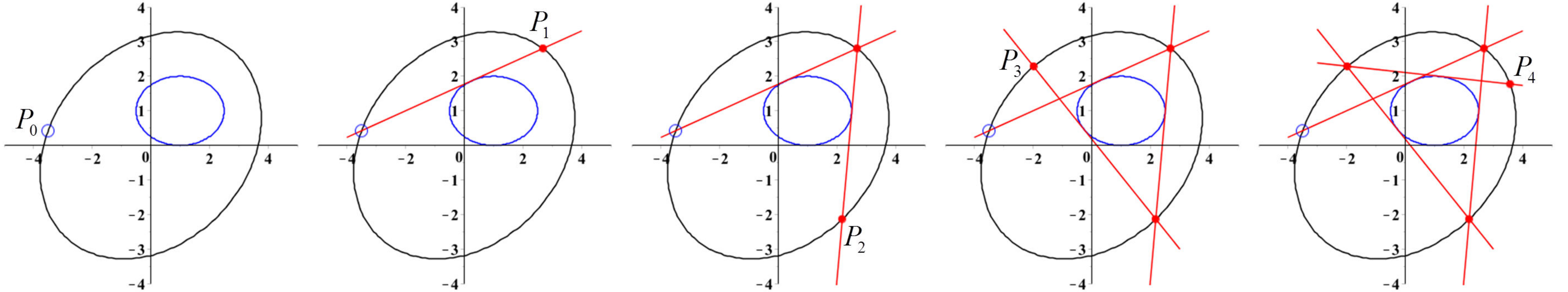
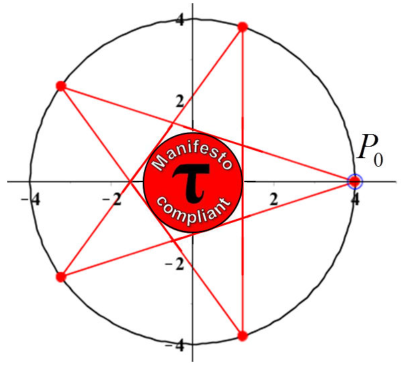
THEOREM OF THE DAY

Poncelet's Porism *Suppose that two ellipses lie in the Euclidean plane, with one totally enclosed by the other. If a closed n-edge polygonal line may be inscribed in the outer ellipse so as to circumscribe (i.e. its edges being tangent to) the inner ellipse, then every point on the outer ellipse lies on some such closed n-edge polygonal line.*

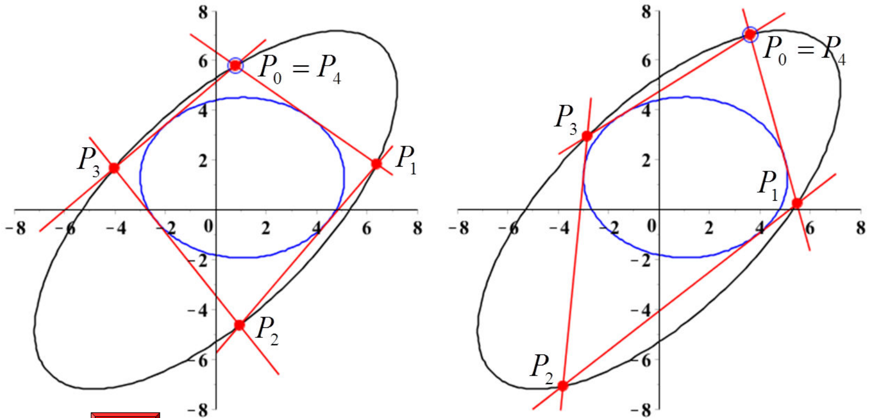


Equivalently, locate a sequence of points $P_0, P_1, \ldots$ on the outer ellipse by specifying a sequence of distinct chords $C_0 = P_0P_1, C_1 = P_1P_2, C_2 = P_2P_3, \ldots$ of the outer ellipse, each chord being tangent to the inner ellipse. Then one of the following occurs, independently of the choice of P_0:

1. The sequence of P_i is infinite;
2. The sequence of P_i terminates with $P_n = P_0$, for some n depending on the ellipses but not on P_0.



In the case where the two ellipses are concentric circles this is obvious because a closed polygonal line may be rotated to touch any point on the outer circle; an example is shown on the right: the five-point star is a closed polygonal line inscribed in an outer circle of radius $r = 4$ and circumscribing a concentric inner circle of radius $r\cos(\tau/5)$. The encircled point on the x axis may be taken as the starting point P_0 for the sequence of chords described above, but any point on the outer circle would give a congruent terminating sequence. Poncelet's remarkable result says that, for arbitrary ellipses, congruence can be sacrificed without losing the invariance of the sequence length. An example is shown below.



Jean-Victor Poncelet was a prisoner of war in Napoleon's Russian campaign when (1813) he proved this theorem ('porism' meaning something like "if it's true in one case then it's true in many or infinitely many"). For Poncelet it was an application of projective geometry; from a modern perspective it is a deep result in algebraic geometry, being essentially equivalent to the fact that a group structure may be given to the set of points lying on an elliptic curve.

Web link: images.math.cnrs.fr/Quand-les-matheux-jouent-au.html (French).

Further reading: *Poncelet's Theorem* by Leopold Flatto, American Mathematical Society, 2008.