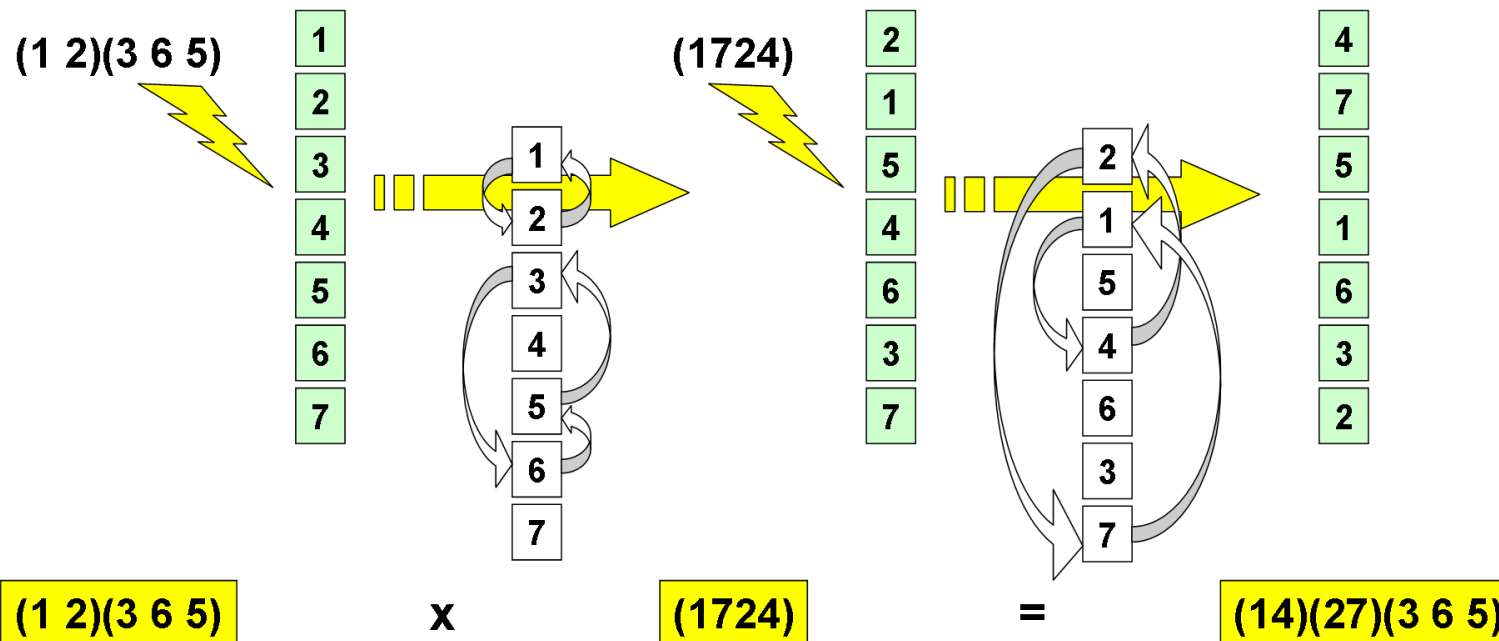




THEOREM OF THE DAY

The Orbit Counting Lemma Let G be a finite permutation group on set Ω . For $g \in G$, let $\text{fix}(g)$ be the number of points of Ω not moved by g . Then

$$\text{number of orbits of } G \text{ on } \Omega = \frac{1}{|G|} \sum_{g \in G} \text{fix}(g).$$



Here $\Omega = \{1, 2, 3, 4, 5, 6, 7\}$. Permutations on Ω can be thought of as rearrangements of the contents of a line of 7 boxes, one for each element of Ω . A combination, or *product*, of two permutations is shown here; the first has $\text{fix}((1\ 2)(3\ 6\ 5)) = 2$, since only the fourth and seventh boxes' contents are unmoved, while $\text{fix}((1\ 7\ 4\ 2)) = 3$. We get a finite permutation group by starting with a collection of permutations, including the *identity* which moves no box's contents, and successively producing all possible finite products. The two permutations used here generate a group of 24 different permutations, each of which can either move the contents of boxes 3, 5 and 6 or the contents of boxes 1, 2, 4 and 7, but without interchanging between these two subsets, which thereby constitute the two orbits of the group. The summation is as follows:

$$\frac{1}{24}(7 + 5 + 3 + 3 + 3 + 5 + 3 + 3 + 4 + 2 + 0 + 0 + 0 + 2 + 0 + 0 + 4 + 2 + 0 + 0 + 0 + 2 + 0 + 0) = \frac{1}{24}(48) = 2.$$

The Orbit Counting Lemma is often attributed to William Burnside (1852–1927). His famous 1897 book *Theory of Groups of Finite Order* perhaps marks its first 'textbook' appearance but the formula dates back to Cauchy in 1845.

Web link: undergraduate.csse.uwa.edu.au/units/CITS7209, see **Lecture 3** (0.3MB).

Further reading: *Permutation Groups* by P.J. Cameron, Cambridge University Press, 1999.

