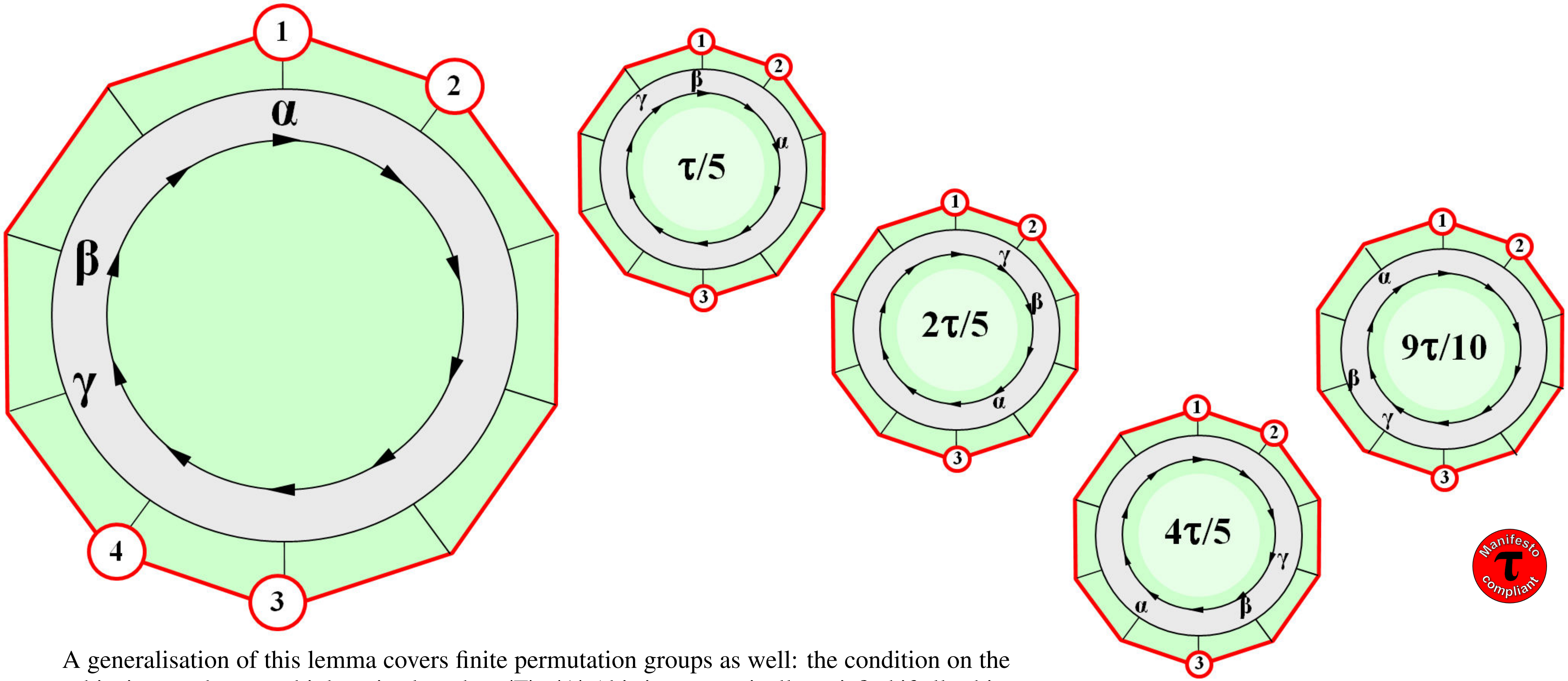
THEOREM OF THE DAY

Neumann's Separation Lemma *Let G be a permutation group acting on an infinite set Ω with no finite orbits. Then for any finite subsets Γ and Δ of Ω there exists a permutation $g \in G$ for which $\Gamma^g \cap \Delta = \emptyset$.*



A generalisation of this lemma covers finite permutation groups as well: the condition on the orbits is now that no orbit has size less than $|\Gamma| \times |\Delta|$ (this is automatically satisfied if all orbits are infinite). We have chosen to illustrate this finite version, with the group G being the rotational symmetries of the 10-point decagon. G acts *transitively*: every point is carried to any other by some rotation so there is only one orbit and this has size 10. On the left, $\Gamma = \{\alpha, \beta, \gamma\}$ and $\Delta = \{1, 2, 3, 4\}$; since $|\Gamma||\Delta| = 3 \times 4 = 12 > 10$, the Separation Lemma is not guaranteed to hold, and indeed it can be checked that no rotation of Γ avoids an overlap with Δ. On the right, however, we have reduced Δ to $\{1, 2, 3\}$ and now the Lemma applies, with both $\Gamma^{3\tau/5} \cap \Delta$ ($\tau = 2\pi$) and $\Gamma^{9\tau/10} \cap \Delta$ being empty.

Peter M Neumann proved his original lemma in 1974 using a 1954 theorem of his father Bernhard Neumann (1909–2002), also a famous group theorist. The extension to the finite case was done in collaboration with Robert G. Burns, Bryan Birch and Sheila Oates Macdonald in the same year.

Web link: cameroncounts.wordpress.com/lecture-notes/synchronization/: a proof of the finite case of the Lemma is given in Lecture 3.

Further reading: *Permutation Groups* by P.J. Cameron, Cambridge University Press, 1999.