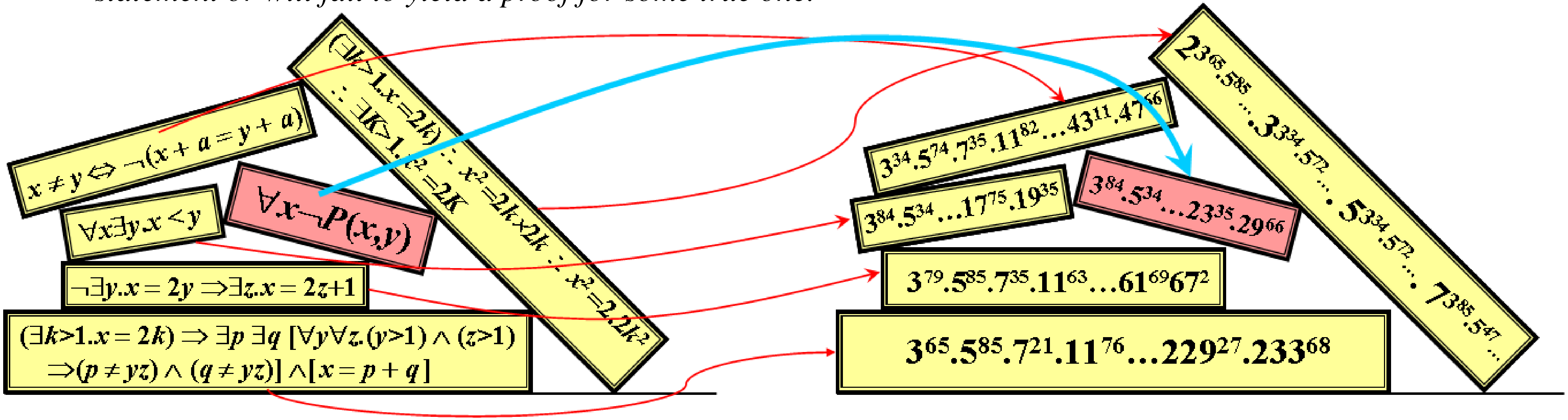
THEOREM OF THE DAY

Gödel's First Incompleteness Theorem *There is no consistent and complete, recursively enumerable axiomatisation of number theory. That is, any such axiomatisation will either yield a proof for some false statement or will fail to yield a proof for some true one.*



"This sentence is false!" Kurt Gödel had a genius for turning such philosophical paradoxes into formal mathematics. In a *recursively enumerable axiomatisation*, T, all sentences — statements and proofs of statements — can, in principle, be listed systematically, although this enumeration will never end, since the list is infinite. This idea was captured by Gödel by giving each sentence s a unique number, denoted $\ulcorner s \urcorner$ and now called a *Gödel number*, a product of powers of primes. On the right of the picture, pay particular attention to the number $3^{84}.5^{34}.\ldots.23^{35}.29^{66}$. This is a number over five hundred digits long — never mind! It will be taken to represent the first-order predicate on the left: $\forall x \neg P(x, y)$, "for all x, $P(x, y)$ is false," which we will denote $G(y)$. Next, Gödel proved a fixed point result: for any arithmetic predicate $Q(x)$, we can find a number q so that the Gödel number of Q with q as input is again the same number: $\ulcorner Q(q) \urcorner = q$. In particular, for G we can find some g with $\ulcorner G(g) \urcorner = g$.

Now suppose that $P(x, y)$ is actually the two-valued predicate which is true if and only if x is the Gödel number of a sentence proving statement number y. Then $G(g)$ means: "sentence number g has no proof *in our numbering system*". Suppose $G(g)$ is provable within T which, because $\ulcorner G(g) \urcorner = g$, is the same as saying that sentence number g has a proof. But this reveals $G(g)$ to be false, and producing a proof of a falsehood is precisely what is meant by saying that T is not consistent. So now if T *is* consistent we therefore know that $G(g)$ cannot be provable, in other words, sentence number g has no proof — G(g) is true! Conclusion: $G(g)$ is a true statement but one which has no proof.

Gödel's announcement of this theorem, in 1931, instantly and forever banished the notion of mathematics as a complete and infallible body of knowledge; and in particular refuted the efforts of Frege, Hilbert, Russell and others to redefine mathematics as a self-contained system of formal logic.

Web link: plato.stanford.edu/entries/goedel/

Further reading: *An Introduction to Gödel's Theorems* by Peter Smith, Cambridge University Press, 2nd edition, 2013.