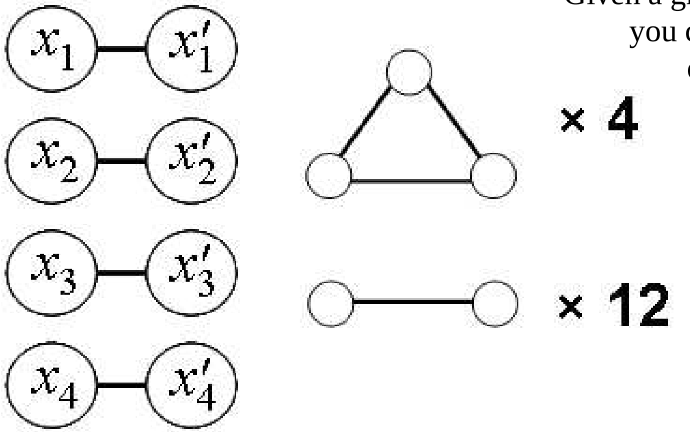
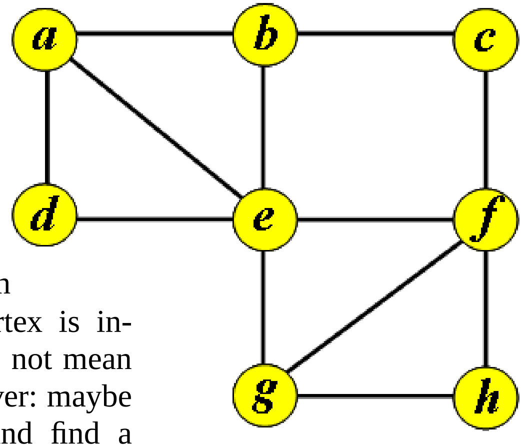
THEOREM OF THE DAY

Karp's Theorem (Detail) *Vertex Cover is NP-Complete.*





Given a graph G, e.g. the one on the right, you can easily find a *vertex cover*: a subset of vertices such that every edge is incident with at least one of these vertices. For this graph the subset $\{b, c, d, e, g, h\}$ is a cover which is *minimal*: every vertex is indispensable. This does not mean that it is a *minimum* cover: maybe you can start again and find a smaller? Certainly, the presence of two disjoint triangles △ means that you cannot do better than 4 vertices because every triangle requires 2 vertices to cover it. So here is the problem known as **Vertex Cover**: given a graph G, and a positive integer K, can you find a cover of size at most K? In our case the target might be set at $K = 4$. This is a **decision problem**: the answer is Yes or No.

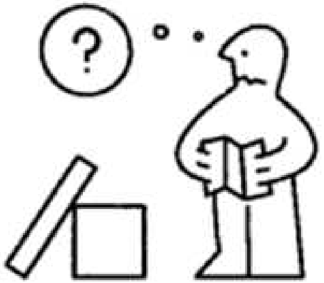
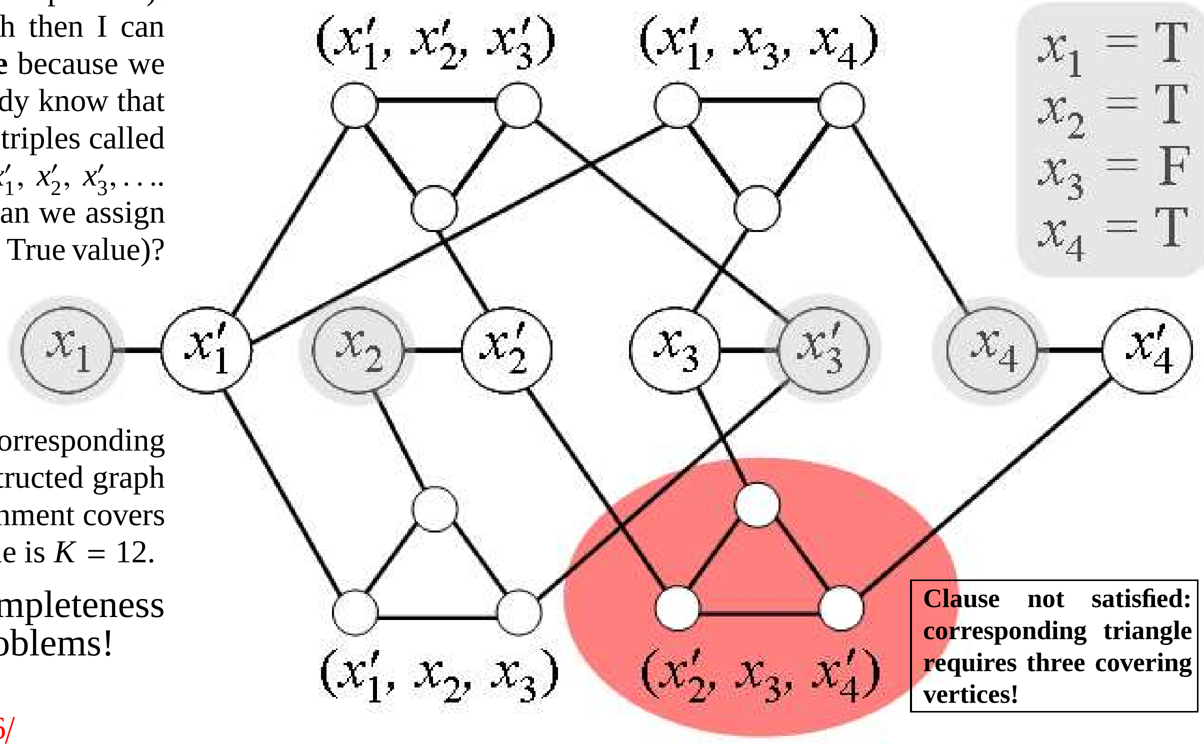


Image courtesy of IKEA

A decision problem is said to belong to the class **NP**, roughly speaking, if evidence for a Yes solution can be checked easily (in a number of steps that is a polynomial function of the input size). Thus, if you assert that $\{a, c, e, g\}$ is a cover of size 4 for our graph then I can quickly spot that edge fh is not covered. Vertex Cover is **NP-complete** because we can transform 3-SAT problems to Vertex Cover problems, and we already know that 3-SAT is **NP**-complete (Cook, 1971). In 3-SAT we have a collection of triples called *clauses* containing logic variables, $x_1, x_2, x_3, \ldots$, and their negations $x_1', x_2', x_3', \ldots$. If x_i is True then x_i' is False and vice-versa. The decision problem is: can we assign truth values to each x_i so that each clause is *satisfied* (contains at least one True value)? For example, (x_1', x_2', x_3'), (x_1', x_3, x_4), (x_1', x_2, x_3), (x_2', x_3, x_4'), is satisfied by $x_1 = F$, $x_3 = T$, with arbitrary values for x_2 and x_4. How is this instance of 3-SAT transformed into Vertex Cover? We take a triangle for each clause and a single edge for each pair x_i, x_i', as shown above. A further 12 linking edges join clause entries to their corresponding single edge vertices, as shown on the right. The resulting cleverly constructed graph has a cover with just 2 vertices per triangle if and only if a 3-SAT assignment covers the third linking edge to each triangle. The target K value in our example is $K = 12$.



A classic 1972 theorem of Richard Karp asserts the **NP**-completeness of Vertex Cover and no fewer than twenty other decision problems!

Web link: cse312wi12.wordpress.com/2012/03/06/

Further reading: *The Nature of Computation* by Christopher Moore and Stephan Mertens, Oxford University Press, 2011.