



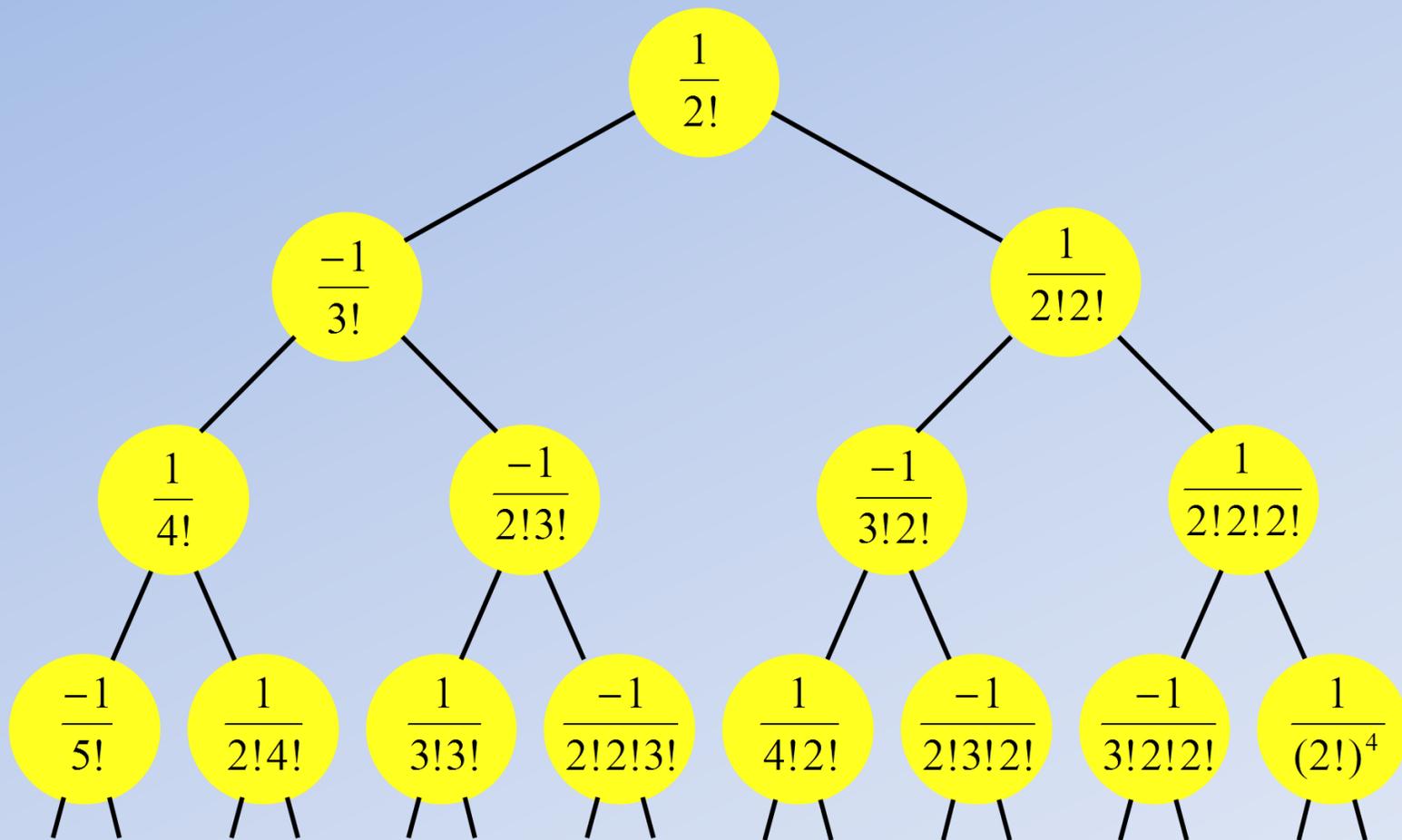
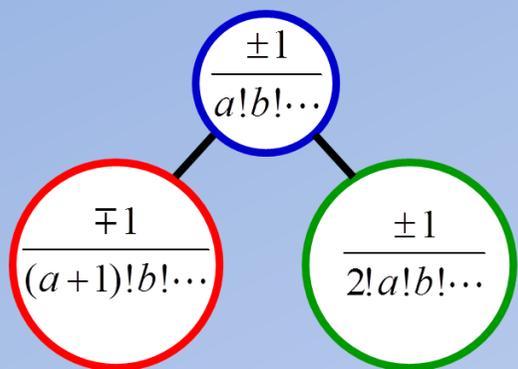
THEOREM OF THE DAY



Euler's Even Zeta Formula For even positive integers the value of $\zeta(s) = \sum_{k=1}^{\infty} 1/k^s$, the Riemann zeta function, is given by

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} \tau^{2n}}{2(2n)!},$$

where B_{2n} is the $2n$ -th Bernoulli number.



A clever scheme due to Max Woon (1997) computes the Bernoulli numbers using the binary tree defined above, starting with root $1/2!$ and growing the tree to left and right as indicated: the left child reverses the sign and increases the left-most factorial argument by 1; the right child preserves sign and prefixes the factorials in the denominator with a $2!$. Now B_n , $n \geq 1$, may be calculated as $n! \times$ the sum of the vertices in the n -th level of the resulting tree. This is shown on the right: the root is level 1, adopting the convention $B_1 = +1/2$, so that B_2 is $2!(-1/3! + 1/(2!)^2) = 1/6$.

The Bernoulli numbers indirectly determine sums of n -th powers of consecutive integers (via Faulhaber's formula); for even powers they directly determine the infinite series sum of reciprocals of powers: $\zeta(2n) = 1 + 1/2^{2n} + 1/3^{2n} + \dots$. Euler's formula cancels the multiplying factorial of Woon's calculation so we may derive $\zeta(2n)$ as $(-1)^{n+1} \tau^{2n} / 2 \times$ sum across level $2n$ in the binary tree. Thus for $2n = 2$ we have $1 + 1/2^2 + 1/3^2 + \dots = \zeta(2) = (+\tau^2/2)(-1/3! + 1/(2!)^2) = \tau^2/24$, which is Euler's celebrated solution to the 'Basel Problem'.

Within five years of his 1734 solution to the Basel Problem Euler had extended it to all even integer powers. He wrestled productively but unsuccessfully with the odd powers question: $B_{2n+1} = 0$ for $n \geq 1$, so no counterpart to his formula is possible; and indeed $\zeta(2n + 1)$ remains shrouded in mystery to this day.

Web link: arxiv.org/abs/1709.09326

Further reading: *The Genius of Euler* by William Dunham (ed.), MAA, 2007.

