THEOREM OF THE DAY

Germain's Theorem *If p is an odd prime such that $2p + 1$ is also prime and if x, y and z are integers none of which is divisible by p then $x^p + y^p \neq z^p$. Such x, y and z cannot therefore be counterexamples to Fermat's Last Theorem for exponent p.*

1	2	3	4	?	6	?	8	9	10
?	12	?	14	15	16	?	18	?	20
21	22	?	24	25	26	27	28	?	30
?	32	33	34	35	36	?	38	39	40
?	42	?	44	45	46	?	48	49	50
51	52	?	54	55	56	57	58	?	60
?	62	63	64	65	66	?	68	69	70
?	72	?	74	75	76	77	78	?	80
81	82	?	84	85	86	87	88	?	90
91	92	93	94	95	96	?	98	99	100

1	2	3	4	5	6	?	8	9	10
11	12	?	14	15	16	?	18	?	20
21	22	23	24	25	26	27	28	29	30
?	32	33	34	35	36	?	38	39	40
41	42	?	44	45	46	?	48	49	50
51	52	53	54	55	56	57	58	?	60
?	62	63	64	65	66	?	68	69	70
?	72	?	74	75	76	77	78	?	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	?	98	99	100

Above left: the state of play regarding Fermat's Last Theorem (FLT) in 1820: integer solutions for $x^n + y^n = z^n$ had been ruled out only for $n = 3$ and 4. The latter case was due to Fermat himself (and was apparently the only proof appearing anywhere in his written work!) while the proof for $n = 3$ is considered due to Leonhard Euler in 1770 (his proof contains an error but his work elsewhere remedies the deficiency). Observe, however, that a solution x, y, z for composite $n = pq$ supplies a solution x^q, y^q, z^q for $n = p$ so, in fact, the odd primes, together with $n = 4$, will settle the whole question. These are the question marks in the grid, as far as $n = 100$.

In the early 1820s Sophie Germain dramatically changed this picture by providing the first real breakthrough on FLT since Fermat's death in 1665. Her work showed that it could be solved in two stages:

Case I: eliminate solutions x, y and z with none being a multiple of n;
Case II: eliminate solutions having one multiple of n,

(two multiples implies all three of x, y and z divide by n and this factor may be removed, reducing back to Case I or II). Germain's Theorem is a powerful condition for Case I to apply, as illustrated by the amber squares in the right-hand grid. In fact, her full theorem is even more powerful than what is stated above, whereby she turned all the red squares amber, the seventy-year old Legendre, with whom she corresponded, continuing this up to $n = 197$. For all these values, FLT had been reduced to Case II.

Sophie Germain (1776 – 1831), who also did pioneering work in the mathematics of elasticity, wrote on equal terms to Gauss and Adrien-Marie Legendre by disguising her sex behind the pseudonym Monsieur Antoine LeBlanc. 'Germain primes' and their application remain a topic of research today, despite the eventual ascent of the FLT mountain by a different face.

Web link: www.agnesscott.edu/lriddle/women/germain-FLT/SGandFLT.htm

Further reading: *Mathematical Expeditions: Chronicles by the Explorers* by R.C. Laubenbacher and D. Pengelley, Springer New York, 2000, chapter 4.