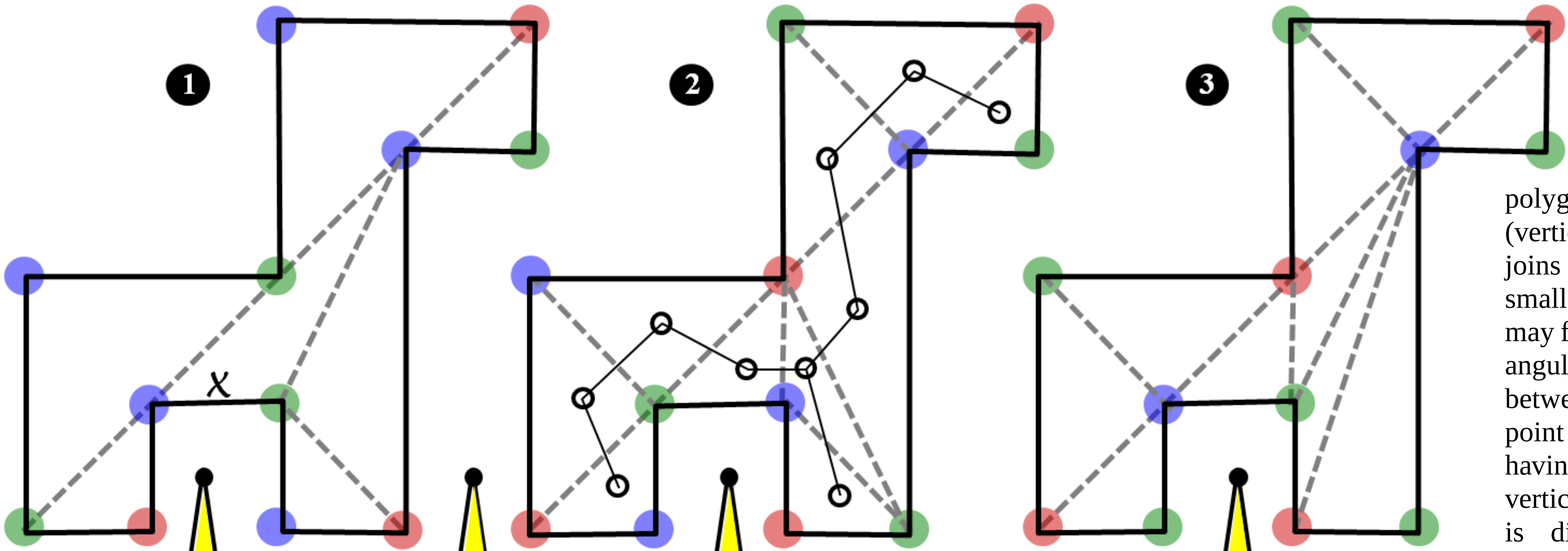
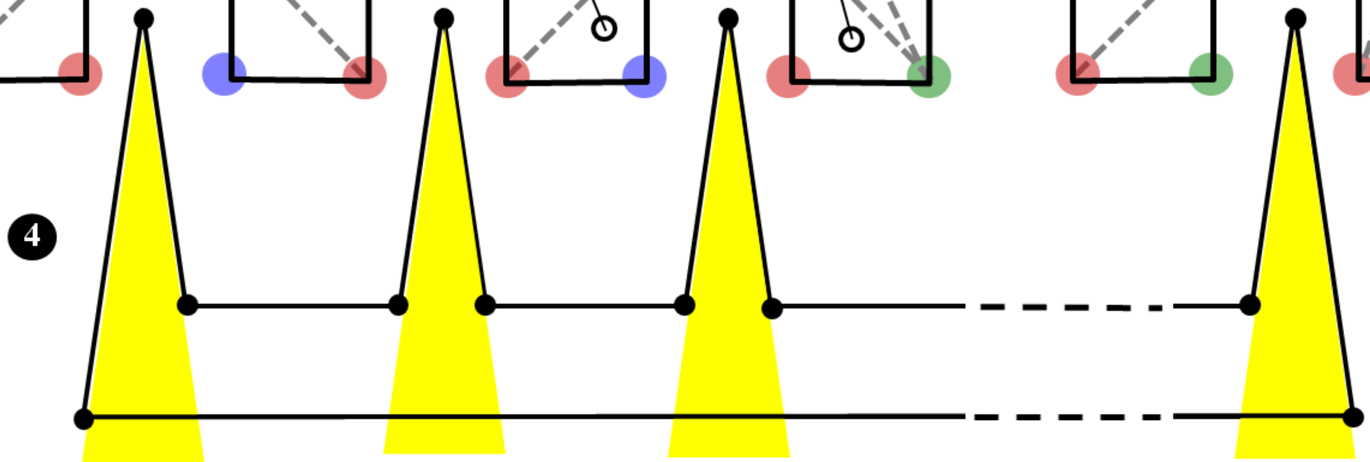
THEOREM OF THE DAY

The Art-Gallery Theorem *Let P be the subset of the Euclidean plane consisting of an n-vertex simple polygon and its interior. Then P contains a finite subset S, of cardinality at most* $\lfloor n/3 \rfloor$*, such that every point of P is joined to some point of S by a straight line contained in P.*





The examples on the left are based on Steve Fisk's beautiful 1978 proof of this theorem. If a triangulation of the polygon is properly 3-coloured (vertices coloured so that no edge joins the same colours) then any smallest colour class of vertices may form our set S. Note that 'triangulation' means adding edges between vertices so that every point of P belongs to a triangle having precisely three polygon vertices. Thus, at ❶, although P is divided up into triangles, the triangulation is incomplete; and although the 3-colouring is valid it does not guarantee a valid set S: no vertex from the smallest colour class, the red vertices, can 'see' the point in P marked 'X'. At ❷, a complete triangulation is given. The 3-colouring is produced systematically by joining the triangles into a tree, as shown, and then traversing the tree 'depth first', 3-colouring triangle by triangle. And indeed each colour class is a valid candidate for set S and has cardinality $\lfloor 12/3 \rfloor = 4$. However, not all triangulations are equal! The one at ❸ produces a set S (the blue vertices) which is optimal, having cardinality 2. So we can sometimes do better than $\lfloor n/3 \rfloor$; but not always—in the example at ❹ each triangle necessarily adds an extra point to set S.

This theorem was published by Václav Chvátal in 1975 in response to a question by Victor Klee. The lower bound can, in the words of Chvátal's original paper "be interpreted as the minimum number of guards required to supervise any art gallery with n walls."

Web link: www.ams.org/samplings/feature-column/fcarc-diagonals1. See www.ams.org/samplings/feature-column/fcarc-klee for historical context.

Further reading: *Discrete and Computational Geometry*, by Satyan L. Devadoss and Joseph O'Rourke, Princeton University Press, 2011, chapter 1.