Additional notes to Theorem 237: Sylvester's Catalecticant

For a binary form $f(x, y)$ of degree n, we are interested in r, the least number of n-th powers of linear factors $(p_i x + q_i y)$ whose sum is f. Zach Teitler provided me with the following very helpful gloss on what can happen:

> Sylvester's theorem [as written down by Sylvester in 19th century language] is not really a correct theorem in the current sense. There are forms $f(x, y)$ that have $r > (n + 2)/2$. For example, the form $f(x, y) = xy^{n-1}$ has degree n, and $r = n$. The counterexamples to Sylvester's 'theorem' are a set of Lebesgue measure zero. I think that Sylvester knew this, and implicitly intended his theorem to mean "usually" or "almost certainly" or something like that.
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> It's true that, in the equation $f(x, y) = (p_1 x + q_1 y)^n + \ldots$, counting the number of unknowns p_i, q_i, and the number of equations (coefficients of $x^j y^{n-j}$ to be matched on the two sides) leads to the conclusion that if $r \geq (n + 2)/2$ then there are at least as many unknowns as equations, so it is an underdetermined system and there "should" be a solution. But there may be bad luck, the system of equations might have no solution — it might be a degenerate system.
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> What you can do is form the catalecticant:
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> - if n is even, it is square, $\frac{n+2}{2} \times \frac{n+2}{2}$,
> - if n is odd, it is size $\frac{n+1}{2} \times \frac{n+3}{2}$, i.e., "off by one" from being square,
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> a Hankel matrix whose (i, j) entry is the a_{i+j} coefficient of f (indexing from zero). And say h is the rank of this matrix. Then it turns out that either $r = h$, or $r = n + 2 - h$ — here r =number of powers needed, n =degree of $f(x, y)$. "Generally" (in the sense of "with probability 1") it is $r = h$, but there is a measure zero set with $r = n + 2 - h$. Exception, when $h = 1$, then $r = 1$. There is no $r = n + 2 - 1 = n + 1$, that doesn't occur. But $h = 2, r = n + 2 - 2 = n$ does occur.
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> In particular if n is even and the catalecticant determinant is nonzero, then $h = (n + 2)/2$, and r must be $(n + 2)/2$. If the catalecticant determinant is zero, then $h < (n + 2)/2$, so either $r = h \leq n/2$, or ("with probability zero") $r > (n + 2)/2$.
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> If n is odd then we are dealing with a non-square matrix. It still makes sense to talk about its rank, but the resulting set will not be a hypersurface (primal).
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> A priori, if the system of equations involved in finding the p_i's and q_i's is degenerate, then it has no solution and it's not immediately clear that adding more terms (increasing r, adding more p's and q's) will fix that. But it turns out that it does. I don't know a slick explanation in terms of equations. One idea is write down an expression for every monomial $x^a y^b = (p_1 x + q_1 y)^{a+b} + \ldots$, so every monomial can be written as a sum of powers, and then every homogeneous polynomial can be written as a sum of powers (just using enough for each term in the homogeneous polynomial). So for every $f(x, y)$ there is *some* sufficiently large r that admits a solution.
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> By the way, for binary forms ($f(x, y)$ with 2 variables) we know that $r \leq n$ always holds and $r = n$ is needed for some $f(x, y)$, i.e., $r = n$ is the largest value of r. But for $f(x, y, z)$ or $f(x_1, .., x_k)$ with $k > 2$ variables, it is wide open — it is not known what is the largest value of r. There are some bounds and a few small cases have been solved. I think the smallest open case is 3 variables and degree 6, where there exist polynomials with $r = 12$, we know $r \leq 18$, and it is open whether the highest value of r is 12, 18, or somewhere in between.

Joseph M. Landsberg drew my attention to a different way of expressing Sylvester's work in modern form:

> There is a basic theorem which says the rank of the Catalecticant matrix tells you the smallest r such that there exists a decomposition over the complex numbers into r n-th powers within ϵ ... There are two issues: 1. being expressible as sums of r powers, and 2. being approximately expressible as sums of r powers. For example $x^{n-1}y$ can be written approximately as the sum of two n-th powers "within ϵ" but its minimal representation is as a sum of n n-th powers. The rank of the catalecticant matrix only detects property 2. See Thm 3.5.2.1 of the tensor book [J.M. Landsberg, *Tensors: Geometry and Applications*, American Mathematical Society, 2012].

The said Thm 3.5.2.1 is in terms of *border rank*: the least r such that a form is a sum of r n-th powers or a *limit* of such sums. I got some more help from Teitler to relate this theorem to the "with probability 1" decomposition referred to above:

> Given a binary form $f = f(x, y)$, the rank of the catalecticant matrix tells you the smallest r such that for every ϵ, there exists some decomposition over the complex numbers into r n-th powers of a form $f' = c_1\ell_1(x, y)^n + ... + c_r\ell_r(x, y)^n$, such that f' is within ϵ of f, in the sense that the vector of coefficients of f' is within ϵ of the vector of coefficients of f.
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> One possibility is that f itself has a decomposition using r powers — so, $f' = f$.
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> If f itself does not have such a decomposition then by taking $\epsilon = 1, 1/2, 1/3, 1/4, \ldots$, we can get a sequence $f'_1, f'_2, f'_3, f'_4, \ldots$, that approaches f. So, f (or rather the vector of coefficients of f) is in the closure of the locus of (vectors of coefficients of) forms that admit r-term decompositions.
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> The "with probability 1" bit means that if we take an epsilon ball around f, ... well, a slight subtlety here.
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> Let me resort to analogy. What does it mean to say that "with probability 1" a point on the earth's surface is not on the equator? If X is a point on the Earth's surface, an epsilon ball around X has almost all points off of the equator, but for that matter it has almost all points off of the earth's surface too. What we mean is more like: the intersection of that ball, with the earth's surface, using some kind of induced measure on the earth's surface, has a zero measure set on the equator and a full measure set off of the equator. One can imagine different ways that Lebesgue measure in $\mathbb{R}^3$ could give rise to a reasonable measure on a smooth surface like a sphere. If instead of a sphere we had a singular, non-manifold surface (such as a double cone, where the vertex is a non-smooth point) then it would be more tricky.
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> These "sets of (vectors of coefficients of) forms admitting r-term decompositions", and their closures, are quite singular, unfortunately.
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> But fortunately we are more or less saved by replacing measure with something simpler.
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> The equator is defined on the earth's surface by an equation (say, "$z = 0$"). Any set defined as the set of solution points of a system of polynomial equations is lower-dimensional and so automatically has empty interior, and presumably measure zero in any reasonable sort of measure.
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> And just so — within the set of (vectors of coefficients of) forms admitting r-term decompositions, the subset consisting of "dishonest" points is defined by some equations, so it has empty interior (in the relative topology) and "should" have measure zero in any sort of reasonable measure.