THEOREM OF THE DAY

Cramer's Rule *Let X be an $n \times n$ matrix. Denote by $M_{i,j}$ the determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting row i and column j of X. Let C be the $n \times n$ matrix whose ij-th entry is $(-1)^{i+j}M_{i,j}$. Then in the matrix product of X with the transpose of C all diagonal entries are equal to the determinant of X and all other entries are zero. That is, we have the identity $XC^T = I_n \det X$.*

Cramer's Rule is an elegant explicit way to specify the inverse of a nonsingular matrix. But we will apply it to demonstrate a curious geometrical fact. From the plot on the right define X to be the following 4×4 matrix:

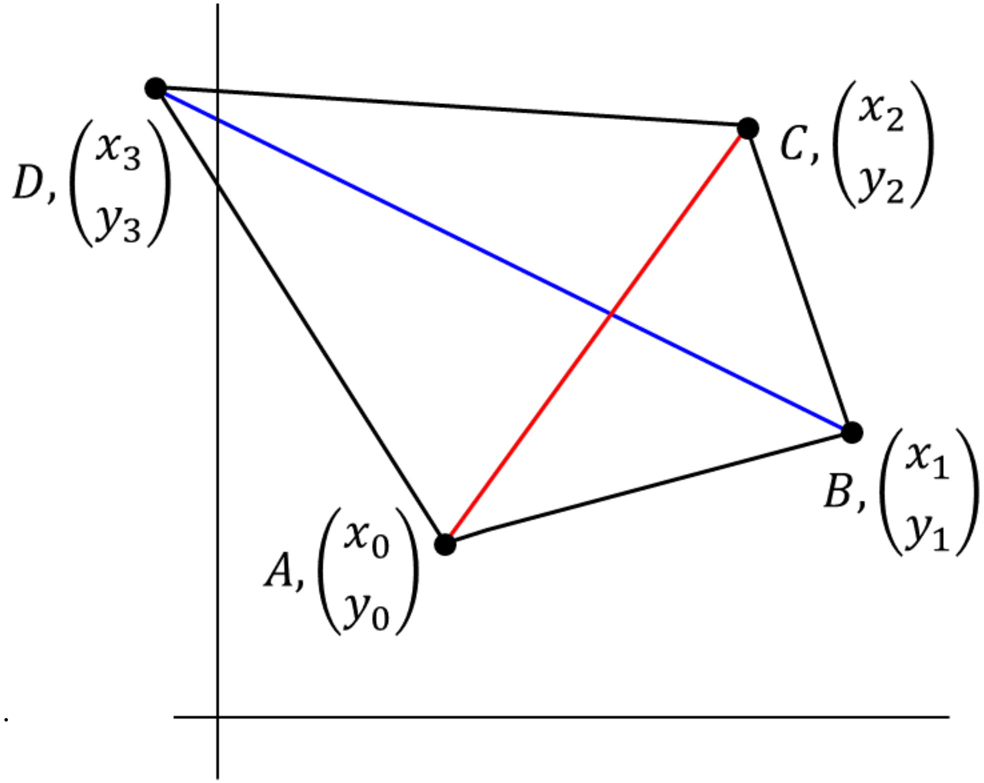
$$X = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}.$$

Now vector algebra tells us that the values of $M_{1,j}$, for $j = 1,2,3,4$ are twice the areas of the triangles on each subset of three of the points A, B, C, D. For example,

$$M_{1,1} = 1 \times (x_2y_3 - x_3y_2) - 1 \times (x_1y_3 - x_3y_1) + \times (x_1y_2 - x_2y_1)\,,$$

which is twice the area, denote this $\frac{1}{2}|BCD|$, of triangle BCD (the deleted column is the missing point). Now consider the product of Cramer's Rule:

$$XC^T = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix} \times \begin{pmatrix} M_{1,1} & & \\ -M_{1,2} & \dots & \\ M_{1,3} & & \\ -M_{1,4} & & \end{pmatrix} = \begin{pmatrix} \det X & 0 & 0 & 0 \\ 0 & \det X & 0 & 0 \\ 0 & 0 & \det X & 0 \\ 0 & 0 & 0 & \det X \end{pmatrix}.$$



The zero entry in row 2, column 1 tells us that $M_{1,1} - M_{1,2} + M_{1,3} - M_{1,4} = 0$. That is, the alternating sum of the four triangle areas above right is zero. This is not a surprise because it is equivalent to saying that $|BCD| + |ABD| = |ACD| + |ABC|$. However, the zero entry in row 3, column 1 says that we still get zero if the horizontal coordinate of each triangle's missing point multiplies the triangle's area. Thus, $x_0|BCD| + x_2|ABD| = x_1|ACD| + x_3|ABC|$. This is perhaps more surprising. (Note that the first row of our matrix has been added merely to turn the 3×4 triangle area calculation into a 4×4 matrix; however, you may discover that it offers some additional geometric information.)

Gabriel Cramer, a Swiss mathematician contemporary with Leonhard Euler, invented this rule in the form of a method for writing down explicit solutions of sets of linear equations. It first appeared as an appendix to Cramer's influential 1750 text *Introduction à l'analyse des lignes courbes algébraique* and has remained textbook material every since.

Web link: www.m-hikari.com/ams/ams-2014/ams-133-136-2014/index.html: see "Old and New Proofs of Cramers Rule" by Maurizio Brunetti.
Further reading: *Introduction to Linear Algebra* by Gilbert Strang, Wellesley-Cambridge Press, 6th edition, 2023, chapter 5.