THEOREM OF THE DAY

Galois' Theorem on Finite Fields *A finite field with n elements exists if and only if $n = p^k$ for some prime p and some non-negative integer k. Moreover this field is unique up to isomorphism.*

×	0	1	2	x	$x+1$	$x+2$	$2x$	$2x+1$	$2x+2$
0	0	0	0	0	0	0	0	0	0
1	0	1	2	x	$x+1$	$x+2$	$2x$	$2x+1$	$2x+2$
2	0	2	1	$2x$	$2x+2$	$2x+1$	x	$x+2$	$x+1$
x	0	x	$2x$	2	$x+2$	$2x+2$	1	$x+1$	$2x+1$
$x+1$	0	$x+1$	$2x+2$	$x+2$	$2x$	1	$2x+1$	2	x
$x+2$	0	$x+2$	$2x+1$	$2x+2$	1	x	$x+1$	$2x$	2
$2x$	0	$2x$	x	1	$2x+1$	$x+1$	2	$2x+2$	$x+2$
$2x+1$	0	$2x+1$	$x+2$	$x+1$	2	$2x$	$2x+2$	x	1
$2x+2$	0	$2x+2$	$x+1$	$2x+1$	x	2	$x+2$	1	$2x$

+	0	1	2	x	$x+1$	$x+2$	$2x$	$2x+1$	$2x+2$
0	0	1	2	x	$x+1$	$x+2$	$2x$	$2x+1$	$2x+2$
1	1	2	0	$x+1$	$x+2$	x	$2x+1$	$2x+2$	$2x$
2	2	0	1	$x+2$	x	$x+1$	$2x+2$	$2x$	$2x+1$
x	x	$x+1$	$x+2$	$2x$	$2x+1$	$2x+2$	0	1	2
$x+1$	$x+1$	$x+2$	x	$2x+1$	$2x+2$	$2x$	1	2	0
$x+2$	$x+2$	x	$x+1$	$2x+2$	$2x$	$2x+1$	2	0	1
$2x$	$2x$	$2x+1$	$2x+2$	0	1	2	x	$x+1$	$x+2$
$2x+1$	$2x+1$	$2x+2$	$2x$	1	2	0	$x+1$	$x+2$	x
$2x+2$	$2x+2$	$2x$	$2x+1$	2	0	1	$x+2$	x	$x+1$

A *field* is a collection of 'numbers' with addition and multiplication defined on it so as to behave analogously to the reals or rationals: there is a 0 for addition, a 1 for multiplication, you can divide consistently, etc. When the number of elements of the field is *n*, a non-negative integer, it is called *finite*. If *n* is a prime then addition and multiplication modulo *p* defines a field, denoted GF(*p*). Thus in GF(3), $2 \times 2 = 2 + 2 = 1$ since $4 \pmod 3 = 1$. But in, say GF(3^2), modulo arithmetic on its own will not do, since $3 \times 3 = 0 \pmod 9$ and 'properly behaved' arithmetic should not give zero as the product of two nonzero numbers. Instead the 'numbers' of GF(p^k) are the polynomials of degree $k - 1$ over GF(*p*), with addition modulo *p* and multiplication defined using remainders modulo some chosen *irreducible polynomial*: one which cannot be written as a product of two polynomials of lower degree.

In the example above, GF(9) is constructed using the polynomial $x^2 + 1$, irreducible over GF(3). Thus x^2 can be written as $(x^2 + 1) + 2$ modulo 3, so the remainder is 2 and, in GF(9), $x^2 = 2$. Now we can calculate, say, $(2x + 1) \times (x + 2) = 2x^2 + 5x + 2 = 2 \times 2 + 5x + 2 = 5x + 6 = 2x$ modulo 3. The addition table is more easily calculated and divides naturally into regions, shown in colours here, which repeat according to the addition table of the base field *p*. By the way, in GF(9) this reminds one of a Sudoku puzzle, and one can indeed construct one. Replace x by 3 in the addition table. Add 0 to the first row of each 3×3 block; add 3 to the second row; and add 6 to the final row. Take remainders modulo 9 and add 1. The result is a completed Sudoku.

The uniqueness of the finite field of order p^k is actually due to the American mathematician Eliakim Moore in 1893. Evariste Galois invented large parts of modern algebra, including finite field theory, before dying in a duel in 1832 at the age of 20. The great pianist Alfred Brendel has said he can forgive the early death of Franz Schubert "as little as I can forgive the death of Bücher, Masaccio or Keats, all of whom died even younger." The loss of Galois, youngest of all, is surely no less unforgiveable.

Web link: www.maths.tcd.ie/pub/Maths/Courseware/FiniteFields/GF.pdf (600KB pdf)

Further reading: *Galois Theory* by Ian Stewart, Chapman & Hall/CRC; 5th edition, 2022. The Brendel quote is in The Veil of Order, Faber, 2002, p.122.