

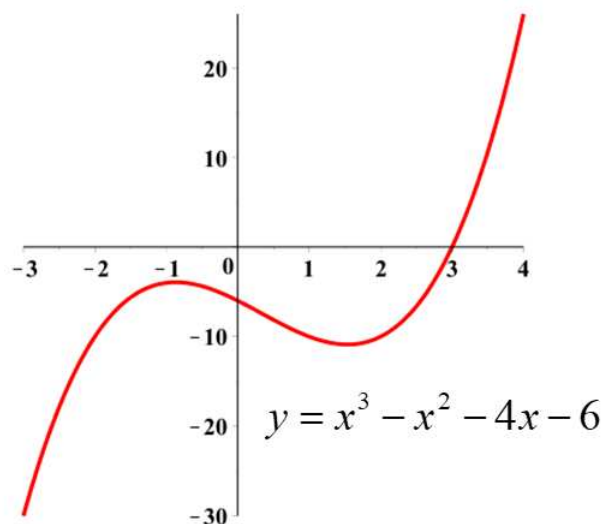


# THEOREM OF THE DAY

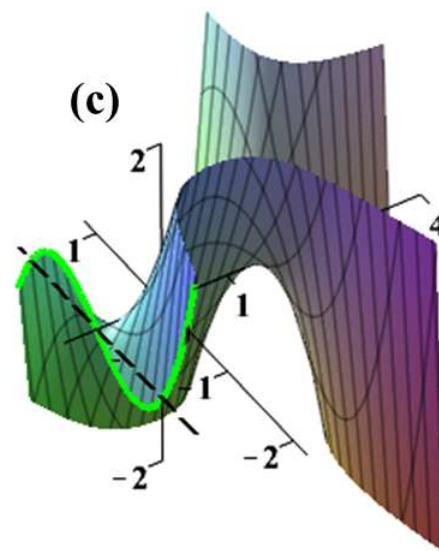
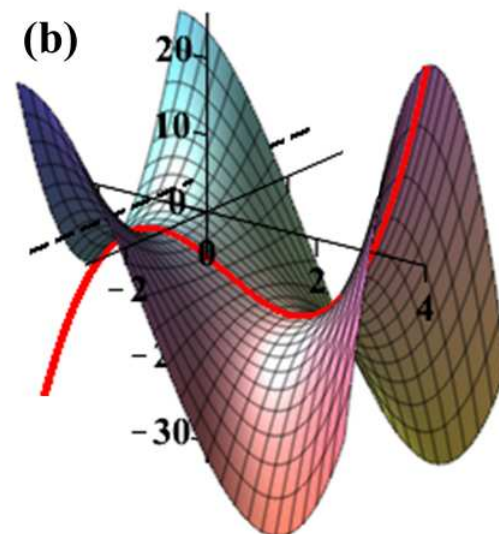
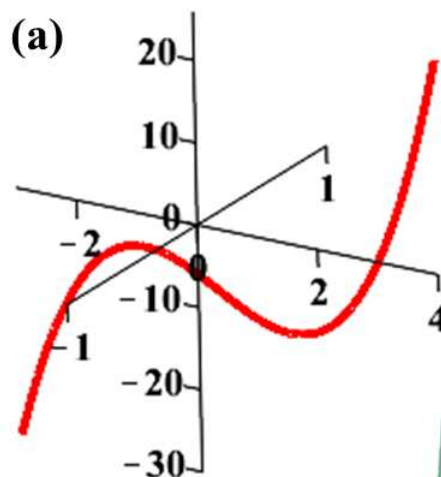
**The Fundamental Theorem of Algebra** *The polynomial equation of degree  $n$ :*

$$z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0,$$

with  $n \geq 1$  and with the  $a_i$  belonging to  $\mathbb{C}$ , the complex numbers, has a solution in  $\mathbb{C}$ . As a consequence, the polynomial can be factorised as  $(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$ , where the  $\alpha_i$  are again in  $\mathbb{C}$  and are precisely the roots of the polynomial.



We begin our story with a 2-dimensional plot of a polynomial curve in one variable,  $y = f(x)$ . Our choice of  $f(x)$ , above left, is the cubic expression  $x^3 - x^2 - 4x - 6$ . It has a single real root,  $x = 3$ : a value for which  $f(x) = 0$ , with the curve meeting the horizontal axis. But we can picture the same curve living in three dimensions, as shown at (a), with a second horizontal axis at right angles to the real number line. This represents all real number multiples of  $i$ , the imaginary number whose square is  $-1$ . Together, the two axes define a plane, called the complex plane. We may write  $f(z)$  instead of  $f(x)$  to indicate that our variable is now a complex number,  $z = a + ib$ . Across the complex plane,  $f$  takes complex values: to plot them would require two more dimensions! Instead we plot the real and imaginary parts of  $f(z)$  separately. At (b) we see that, at  $z = -1 \pm i$ , the surface plotting the real part of  $y = f(z)$  passes up through the complex plane: potentially we have two more roots! And indeed a plot of the imaginary part of  $f(z)$ , shown at (c), confirms that the line  $x = -1$  is indeed crossed by  $f(z)$  at  $\pm i$ .



**Function**

$$f(z) = z^3 - z^2 - 4z - 6$$

**Roots**

$$\alpha_1 = 3, \alpha_2 = -1 + i, \alpha_3 = -1 - i$$

**Factorisation**

$$f(z) = (z - 3)(z + 1 - i)(z + 1 + i)$$

The complex plane is often called the Argand diagram in honour of Jean-Robert Argand (1768–1822) an accountant and amateur mathematician who gave the first full statement and proof of the Fundamental Theorem in 1806 (although a somewhat simpler version had effectively been proved by James Wood in 1798 and by Gauss in his doctoral dissertation of 1797).

**Web link:** [www.maa.org/programs/maa-awards/writing-awards/euler-and-the-fundamental-theorem-of-algebra](http://www.maa.org/programs/maa-awards/writing-awards/euler-and-the-fundamental-theorem-of-algebra)

**Further reading:** *The Fundamental Theorem of Algebra* by B. Fine & G. Rosenberger, Springer, 1997.

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