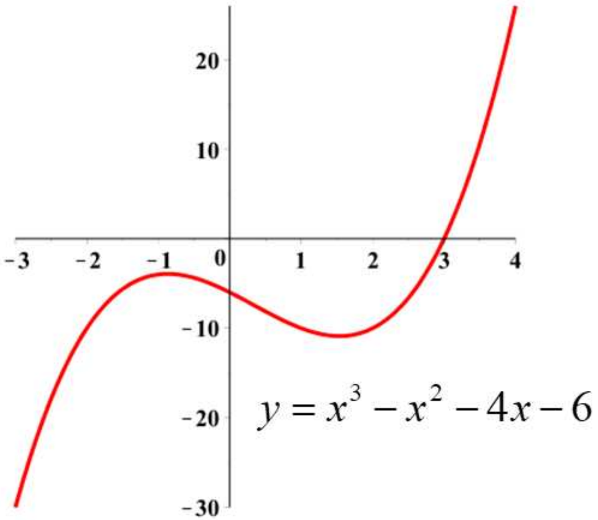
THEOREM OF THE DAY

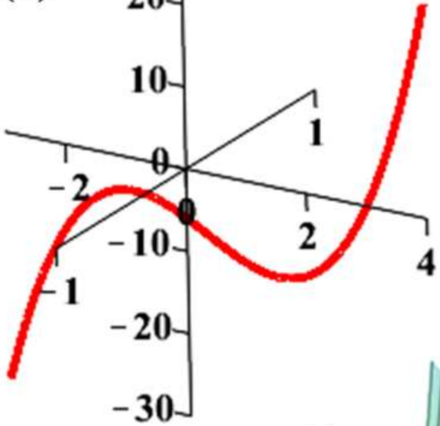
The Fundamental Theorem of Algebra *The polynomial equation of degree n:*

$$z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n = 0,$$

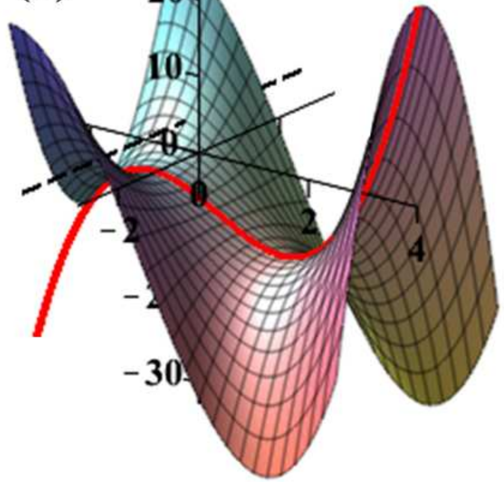
where the a_i belong to $\mathbb{C}$, the complex numbers, has at least one solution in $\mathbb{C}$. As a consequence, the polynomial can be factorised as $(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$, where the α_i are again complex numbers and are precisely the roots of the polynomial.



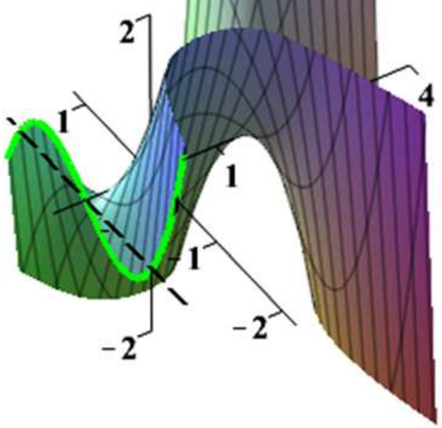
(a)



(b)



(c)



Function
$f(z) = z^3 - z^2 - 4z - 6$
Roots
$\alpha_1 = 3,\ \alpha_2 = -1 + i,\ \alpha_3 = -1 - i$
Factorisation
$f(z) = (z-3)(z+1-i)(z+1+i)$

We begin our story with a 2-dimensional plot of a polynomial curve in one variable, $y = f(x)$. Our choice of $f(x)$, above left, is the cubic expression $x^3 - x^2 - 4x - 6$. It has a single real root, $x = 3$: a value for which $f(x) = 0$, with the curve meeting the horizontal axis. But we can picture the same curve living in three dimensions, as shown at (a), with a second horizontal axis at right angles to the real number line. This represents all real number multiples of i, the imaginary number whose square is -1. Together, the two axes define a plane, called the complex plane. We may write $f(z)$ instead of $f(x)$ to indicate that our variable is now a complex number, $z = a + ib$. Across the complex plane, f takes complex values: to plot them would require two more dimensions! Instead we plot the real and imaginary parts of $f(z)$ separately. At (b) we see that, at $z = -1 \pm i$, the surface plotting the real part of $y = f(z)$ passes up through the complex plane: potentially we have two more roots! And indeed a plot of the imaginary part of $f(z)$, shown at (c), confirms that the line $x = -1$ is indeed crossed by $f(z)$ at $\pm i$.

The complex plane is often called the Argand diagram in honour of Jean-Robert Argand (1768–1822) an accountant and amateur mathematician who gave the first full statement and proof of the Fundamental Theorem in 1806 (although a somewhat simpler version had effectively been proved by James Wood in 1798 and by Gauss in his doctoral disseration of 1797).

Web link: www.maa.org/programs/maa-awards/writing-awards/euler-and-the-fundamental-theorem-of-algebra

Further reading: *The Fundamental Theorem of Algebra* by B. Fine & G. Rosenberger, Springer, 1997.