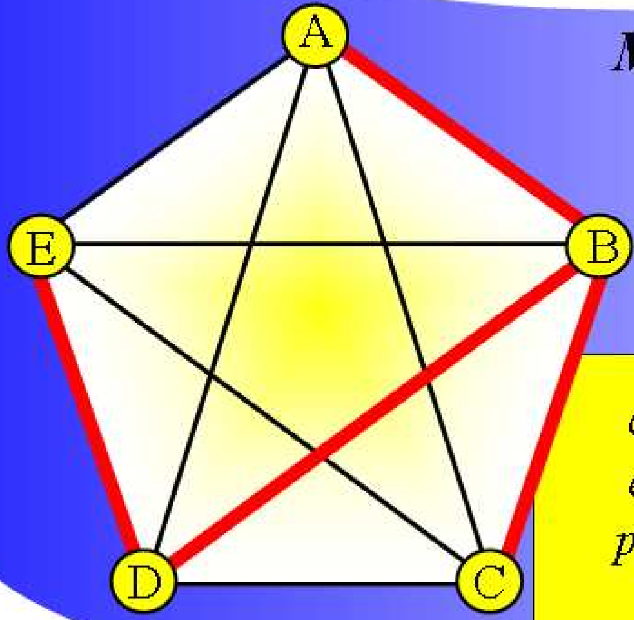
THEOREM OF THE DAY

The Girard–Newton Identities *For a fixed set S of variables, denote by e_k, $0 \le k \le |S|$, the k-th elementary symmetric polynomial in the variables of S; that is $e_k = \sum_{\substack{X\subset S\\|X|=k}} \prod_{x\in X} x$, with $e_0 = 1$. Denote by p_k the k-th power sum over S; that is $p_k = \sum_{x\in S} x^k$. Then the following recurrence holds:*

$$ke_k = \sum_{i=1}^{k}(-1)^{i-1} p_i\, e_{k-i}, \quad \text{for } k \ge 1.$$

"…one can appreciate the view held by some people, that if it isn't related to symmetric polynomials, then it isn't combinatorics!"



M:

	B	C	D	E
B	4	−1	−1	−1
C	−1	4	−1	−1
D	−1	−1	4	−1
E	−1	−1	−1	4

$$c(t) = (t-1)\,(t-5)^3$$
$$e_4 = \text{constant term} = 125$$
$$p_1 = 16,\ p_2 = 76,\ p_3 = 376,$$
$$p_4 = 1876$$

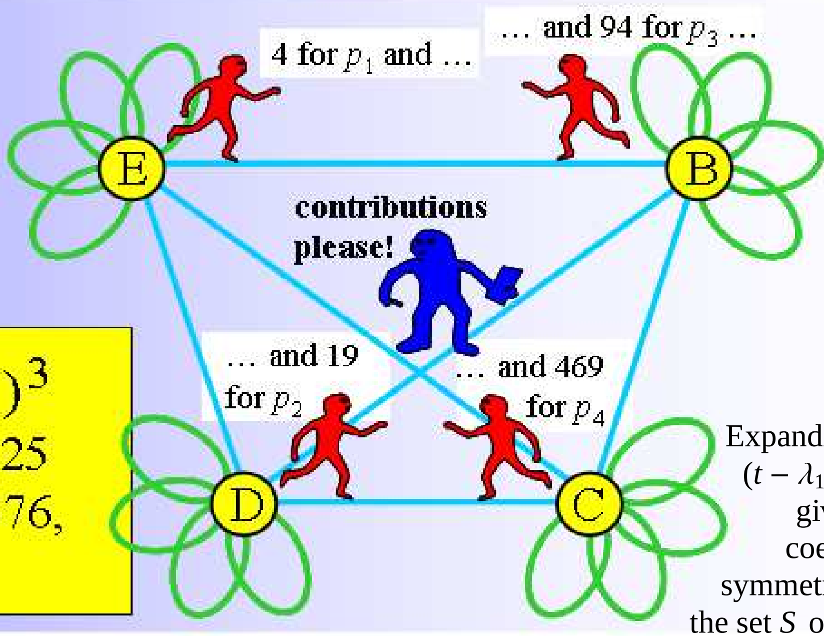


Photo: Adrian Bondy

Expanding, to take an example, $(t-\lambda_1)(t-\lambda_2)(t-\lambda_3)(t-\lambda_4)$, gives a polynomial whose coefficients are elementary symmetric polynomials e_k, with the set S of variables being the set of roots λ_i of the polynomial. These in turn can be written, via the Girard–Newton Identities and back substitution, in terms of power sums of the roots; e.g. the constant term is $\lambda_1\lambda_2\lambda_3\lambda_4 = e_4 = (p_1^4 - 6p_1^2p_2 + 3p_2^2 + 8p_1p_3 - 6p_4)/24$. **An application:** vertex A in the network on the left wishes to discover the number of spanning trees (connected, cycle-free, containing all vertices; e.g. the bold red edges, above left) in the network, *without revealing to anybody their interest in this information*. Via the Matrix Tree Theorem, this number is obtained from the matrix M, centre, top, which records the edges between B, C, D and E (weighted negatively) and their vertex degrees (on the diagonal). In fact, we just calculate the constant term, e_4, of the *characteristic polynomial* $c(t)$ of the matrix. Of course A cannot ask for this information—it would give the game away. But the value of p_k in this case is precisely the sum of the diagonal elements of M^k, which can be obtained thus: the contribution of, say, B is the number of ways B can make a circular walk of k edges in the version of the network on the right, with an odd number of non-loop (negative) edges causing a walk to contribute negatively. So A collects these innocent-seeming, circular walk counts from each vertex, reconstructs the p_k's and, hey presto, counts spanning trees.

These identities were discovered by Isaac Newton, perhaps around 1669, but had been published by Albert Girard in 1629.

Web link: fermatslasttheorem.blogspot.com/2007/02/newtons-identities.html. Some history: mathtourist.blogspot.co.uk/2008/03/.

Further reading: *Combinatorics: Topics, Techniques, Algorithms,* by Peter J. Cameron, CUP, 1994; the quote above right appears at the end of Chapter 13, in connection with Macdonald's *Symmetric Functions and Hall Polynomials*, OUP, 2nd edition, 1998.