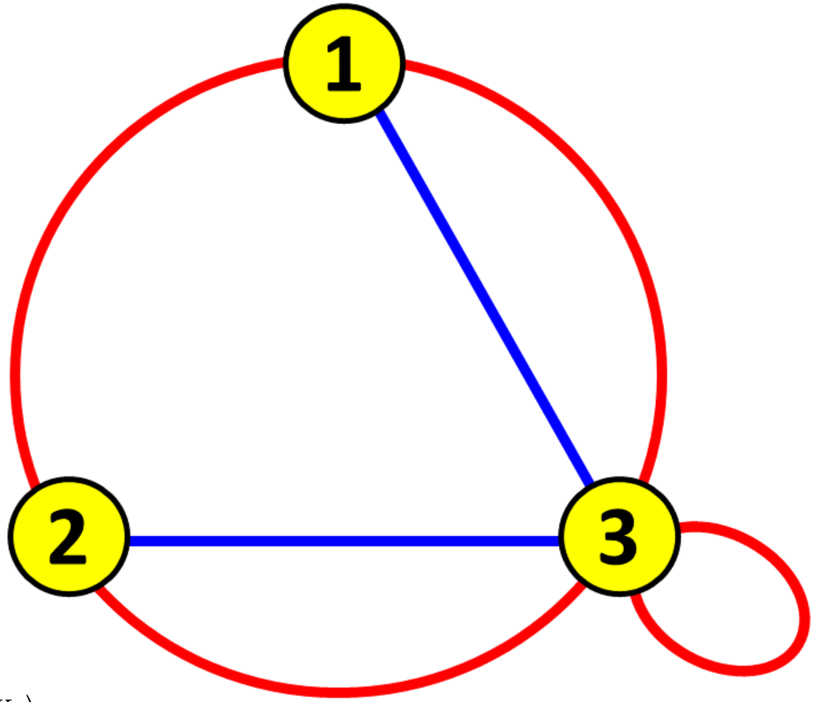
THEOREM OF THE DAY

Schur's Commuting Matrices Bound *Over any field, a collection of linearly independent, mutually commuting $n \times n$ matrices can have cardinality at most $\lfloor n^2/4 \rfloor + 1$.*

The case $n = 2$ is already instructive, even restricting to upper-triangular matrices, where all elements below the main diagonal are zero. Thus, suppose $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and $B = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$ commute. So $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \times \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \times \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. The diagonal entries are $a\alpha$ and $c\gamma$ for both products (since field elements commute), so we need only compare the top right-hand entries, and this comparison rearranges to give: $b(\alpha - \gamma) = \beta(a - c)$. Easy! $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$, for example, satisfy this requirement. The identity matrix $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ automatically qualifies: it commutes with everything. But doesn't this give a total of $3 > \lfloor 2^2/4 \rfloor + 1$ commuting matrices? It does, but they are not linearly independent, since the linear equation $4I_2 - A - B = 0$ is satisfied.

$$X = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$



For general n, construct a set of commuting matrices as follows: consider an $n \times n$ 'template' matrix $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$, where X is $n/2 \times n/2$, if n is even, and is $(n+1)/2 \times (n-1)/2$, if n is odd. Then X has $\lfloor n^2/4 \rfloor$ entries. Choose one of these entries to have value 1 and the rest zero, and observe that these choices give linearly independent matrices whose pairwise products are all the zero matrix. Together with the identity matrix I_n this gives $\lfloor n^2/4 \rfloor + 1$ linearly independent commuting matrices.

In our illustration, two commuting matrices, X and Y, are interpreted as adjacency matrices of two graphs, depicted on the same vertex set as, respectively, red and blue edges. Thus, the 1s in the first row of X specify red edges from vertex 1 to vertices 2 and 3, and so on. Symmetric matrices have been chosen since if there is an edge from vertex u to vertex v then the same edge joins vertex v to vertex u. Now the fact that $XY = YX$ is interpreted in the graph in terms of two-edge walks between pairs of vertices: the number of such walks taking a red edge and then a blue edge is the same as the number taking a blue edge and then a red. For example, there is one red-then-blue walk from 2 to 3: this is red edge 21 followed by blue edge 13. So there must be just one blue-then-red walk: it is blue edge 23 followed by the red loop 33. We can go further: commutativity means, say, $XYX^2 = X^3Y$. So the number of red-blue-red-red walks from 2 to 3 is the same as the number of red-red-red-blue walks (the matrix product tells us this number is 7).

Issai Schur proved this upper bound, and described the collections which achieved it, for matrices of hypercomplex numbers (complex, quaternions, etc, much studied at the end of the 19th century; essentially his proof worked for algebraically closed fields). Nathan Jacobson extended this to arbitrary fields in 1994.

Web link: math.stackexchange.com/questions/2687791

Further reading: *Computational Linear and Commutative Algebra* by Martin Kreuzer and Lorenzo Robbiano, Springer, 2016.