THEOREM OF THE DAY

Wedderburn's Little Theorem *Any finite division ring is commutative.*

$+$	0	1	a	b	c	d	e	f	g
0	0	1	a	b	c	d	e	f	g
1	1	a	0	c	d	b	f	g	e
a	a	0	1	d	b	c	g	e	f
b	b	c	d	e	f	g	0	1	a
c	c	d	b	f	g	e	1	a	0
d	d	b	c	g	e	f	a	0	1
e	e	f	g	0	1	a	b	c	d
f	f	g	e	1	a	0	c	d	b
g	g	e	f	a	0	1	d	b	c

$\times$	1	a	b	c	d	e	f	g
1	1	a	b	c	d	e	f	g
a	a	1	e	g	f	b	d	c
b	b	e	a	f	c	1	g	d
c	c	g	d	a	e	f	b	1
d	d	f	g	b	a	c	1	e
e	e	b	1	d	g	a	c	f
f	f	d	c	e	1	g	a	b
g	g	c	f	1	b	d	e	a

Operations $+$ and $\times$ defined on $S = \{0, 1, a, b, c, d, e, f, g\}$. Cell shading is to facilitate inspection merely.

Take a set S. Combine its elements using a binary operation $*$. Mathematicians have identified properties which $*$ should obey in order to give a 'realistic' arithmetic:

Closed: if x and y are in S then $x * y$ should be too.
Identity: a unique element of S, which we may as well call '1', should be found to be inactive under $*$: that is, $1 * x = x * 1 = x$, for any x.
Inverses: $*$ should be able to reduce anything to 1: given x it should have an inverse $y = x^{-1}$ for which $x * y$ and $y * x$ both give value 1.
Associative law: bracketing should not affect $*$, that is $(x * y) * z$ and $x * (y * z)$ give the same result.
Commutative law: order should not affect $*$: that is $x * y$ and $y * x$ both give the same result.

For a **division ring** we take *two* operations '+' and '×'. Addition should obey *all* the above, although we call the identity '0' instead of '1'. Multiplication is allowed not to be commutative: in the multiplication table above, $b \times c = f$ but $c \times b = d$; if '×' does commute the division ring is called a **field**. Meanwhile, '+' and '×' must interact realistically: $0 \times$ anything gives 0 (so it is customary to omit the first row and column from the '×' table); 0^{-1} will not exist; and a final property should hold:

Distributive law: we can expand brackets: $x \times (y + z) = x \times y + x \times z$ and $(x + y) \times z = x \times z + y \times z$. E.g. using the above tables, $(a + b) \times c = d \times c = b$ and $a \times c + b \times c = g + f = b$.

It is nearly the case that the above tables define a non-commutative (i.e. for '×') division ring which is nevertheless obviously finite! In fact they define a so-called **near-field**, failing in just one respect to obey the hypothesis of our theorem: '+' and '×' are not *left*-distributive. E.g. $b \times (c + d) = b \times e = 1$ but $b \times c + b \times d = f + c = a$.

The surprising discovery that cardinality might influence multiplication was made in 1905 simultaneously by Joseph Wedderburn and Leonard Dickson both, at the time, at the University of Chicago. Our order 9 near-field was also discovered by Dickson in 1905.

Web link: www.theoremoftheday.org/Docs/WedderburnShamil.pdf. The Dickson near-field construction, based on the Galois field GF(9) and having multiplicative group isomorphic to the quaternions, is described at en.wikipedia.org/wiki/Near-field_(mathematics).

Further reading: *Topics in Algebra* by I.N. Herstein, John Wiley & Sons, 2nd edition, 1975.