Theorem of the Day

Wedderburn’s Little Theorem Any finite division ring is commutative.

For a division ring we take two operations ‘+’ and ‘×’. Addition should obey all the above, although we call the identity ‘0’ instead of ‘1’. Multiplication is allowed not to be commutative: in the multiplication table above, \( b \times c = f \) but \( c \times b = d \); if ‘×’ does commute the division ring is called a field. Meanwhile, ‘+’ and ‘×’ must interact realistically: \( 0 \times \text{anything gives 0} \) (so it is customary to omit the first row and column from the ‘×’ table); \( 0^{-1} \) will not exist; and a final property should hold: Distributive law: we can expand brackets: \( x \times (y + z) = x \times y + x \times z \) and \( (x + y) \times z = x \times z + y \times z \). E.g. using the above tables, \( (a + b) \times c = d \times c = b \) and \( a \times b + a \times c = g + f = b \).

It is nearly the case that the above tables define a non-commutative (i.e. for ‘×’) division ring which is nevertheless obviously finite! In fact they define a so-called near-field, failing in just one respect to obey the hypothesis of our theorem: ‘+’ and ‘×’ are not left-distributive. E.g. \( b \times (c + d) = b \times e = 1 \) but \( b \times c + b \times d = f + c = a \).

The surprising discovery that cardinality might influence multiplication was made in 1905 simultaneously by Joseph Wedderburn and Leonard Dickson both, at the time, at the University of Chicago. Our order 9 near-field was also discovered by Dickson in 1905.

Web link: [www.theoremoftheday.org/Docs/WedderburnShamil.pdf](http://www.theoremoftheday.org/Docs/WedderburnShamil.pdf). The Dickson near-field construction, based on the Galois field GF(9) and having multiplicative group isomorphic to the quaternions, is described at [en.wikipedia.org/wiki/Near-field_(mathematics)](http://en.wikipedia.org/wiki/Near-field_(mathematics)).