Take a set \( S \). Combine its elements using a binary operation \( * \). Mathematicians have identified properties which \( * \) should obey in order to give a ‘realistic’ arithmetic:

**Closed:** if \( x \) and \( y \) are in \( S \) then \( x * y \) should be too.

**Identity:** a unique element of \( S \), which we may as well call ‘1’, should be found to be inactive under \( * \): that is, \( 1 * x = x * 1 = x \), for any \( x \).

**Inverses:** \( * \) should be able to reduce anything to 1: given \( x \) it should have an inverse \( y = x^{-1} \) for which \( x * y \) and \( y * x \) both give value 1.

**Associative law:** bracketing should not affect \( * \), that is \( (x * y) * z \) and \( x * (y * z) \) give the same result.

**Commutative law:** order should not affect \( * \); that is \( x * y \) and \( y * x \) both give the same result.

For a division ring we take two operations ‘+’ and ‘\( \times \)’. Addition should obey all the above, although we call the identity ‘0’ instead of ‘1’. Multiplication is allowed not to be commutative: in the multiplication table above, \( b \times c = f \) but \( c \times b = d \); if ‘\( \times \)’ does commute the division ring is called a field. Meanwhile, ‘+’ and ‘\( \times \)’ must interact realistically: 0 \( \times \) anything gives 0 (so it is customary to omit the first row and column from the ‘\( \times \)’ table); \( 0^{-1} \) will not exist; and a final property should hold:

**Distributive law:** we can expand brackets: \( x \times (y + z) = x \times y + x \times z \) and \( (x + y) \times z = x \times z + y \times z \). E.g. using the above tables, \((a + b) \times c = d \times c = b\) and \( a \times c + b \times c = g + f = b \).

It is nearly the case that the above tables define a non-commutative (i.e. for ‘\( \times \)’) division ring which is nevertheless obviously finite! In fact they define a so-called near-field, failing in just one respect to obey the hypothesis of our theorem: ‘+’ and ‘\( \times \)’ are not left-distributive. E.g. \( b \times (c + d) = b \times e = 1 \) but \( b \times c + b \times d = f + c = a \).

The surprising discovery that cardinality might influence multiplication was made in 1905 simultaneously by Joseph Wedderburn and Leonard Dickson both, at the time, at the University of Chicago. Our order 9 near-field was also discovered by Dickson in 1905.

**Web link:** [www.theoremoftheday.org/Docs/WedderburnShamil.pdf](http://www.theoremoftheday.org/Docs/WedderburnShamil.pdf). The Dickson near-field construction, based on the Galois field GF(9) and having multiplicative group isomorphic to the quaternions, is described at [en.wikipedia.org/wiki/Near-field_(mathematics)](http://en.wikipedia.org/wiki/Near-field_(mathematics)).


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