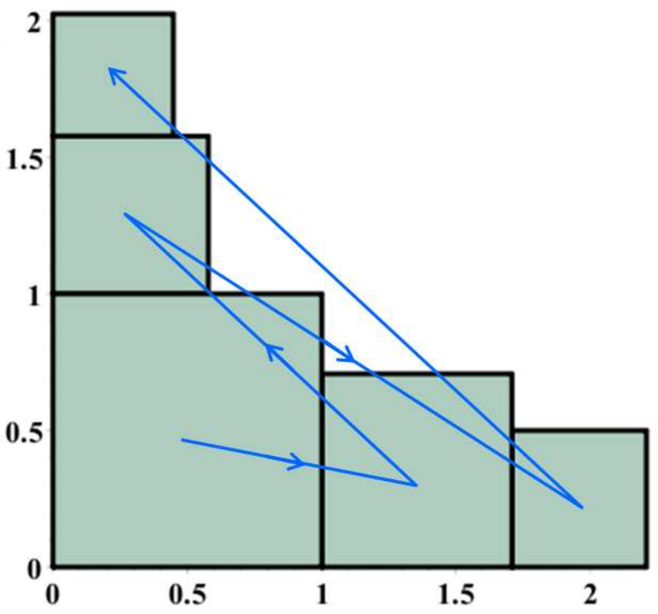
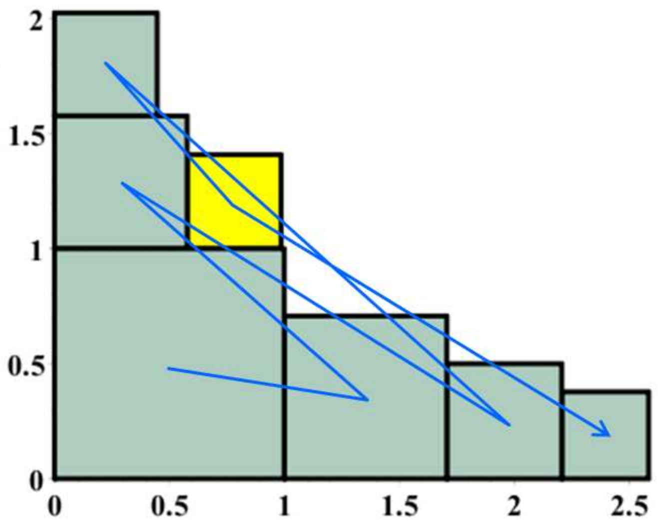
THEOREM OF THE DAY

The Alternating Series Test *If $a_1, a_2, a_3, \ldots$ is a decreasing sequence of positive real numbers (i.e. $0 < \cdots \leq a_3 \leq a_2 \leq a_1$) then the alternating series $a_1 - a_2 + a_3 - \ldots$ converges if and only if the a_n approach zero as n approaches infinity.*

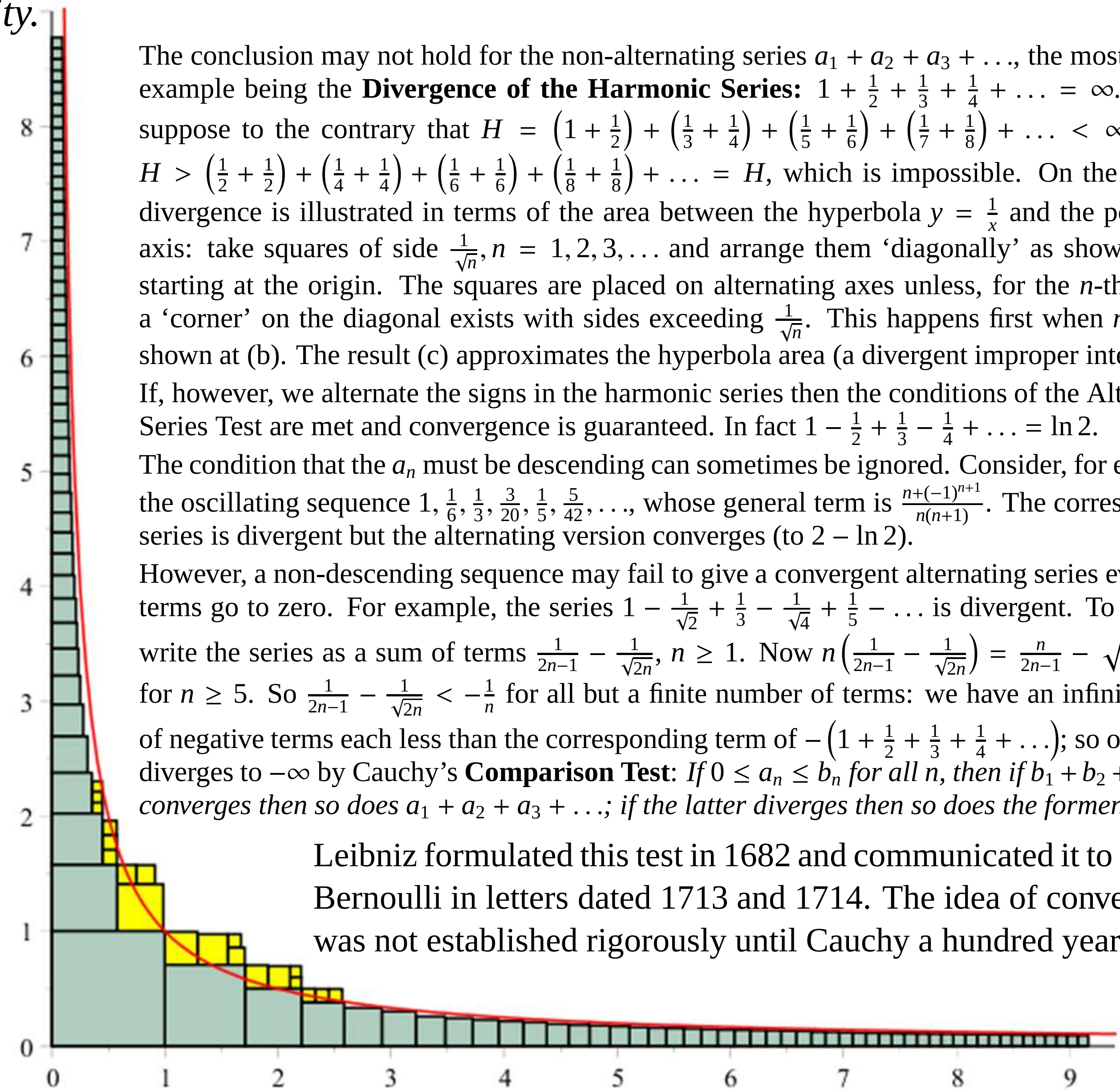
(a)



(b)



(c)



The conclusion may not hold for the non-alternating series $a_1 + a_2 + a_3 + \ldots$, the most famous example being the **Divergence of the Harmonic Series:** $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = \infty$. Indeed suppose to the contrary that $H = \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \left(\frac{1}{7} + \frac{1}{8}\right) + \ldots < \infty$. Then $H > \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{8} + \frac{1}{8}\right) + \ldots = H$, which is impossible. On the left this divergence is illustrated in terms of the area between the hyperbola $y = \frac{1}{x}$ and the positive x axis: take squares of side $\frac{1}{\sqrt{n}}, n = 1, 2, 3, \ldots$ and arrange them 'diagonally' as shown at (a), starting at the origin. The squares are placed on alternating axes unless, for the n-th square, a 'corner' on the diagonal exists with sides exceeding $\frac{1}{\sqrt{n}}$. This happens first when $n = 6$ as shown at (b). The result (c) approximates the hyperbola area (a divergent improper integral).

If, however, we alternate the signs in the harmonic series then the conditions of the Alternating Series Test are met and convergence is guaranteed. In fact $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \ln 2$.

The condition that the a_n must be descending can sometimes be ignored. Consider, for example, the oscillating sequence $1, \frac{1}{6}, \frac{1}{3}, \frac{3}{20}, \frac{1}{5}, \frac{5}{42}, \ldots$, whose general term is $\frac{n+(-1)^{n+1}}{n(n+1)}$. The corresponding series is divergent but the alternating version converges (to $2 - \ln 2$).

However, a non-descending sequence may fail to give a convergent alternating series even if its terms go to zero. For example, the series $1 - \frac{1}{\sqrt{2}} + \frac{1}{3} - \frac{1}{\sqrt{4}} + \frac{1}{5} - \ldots$ is divergent. To see this, write the series as a sum of terms $\frac{1}{2n-1} - \frac{1}{\sqrt{2n}}, n \geq 1$. Now $n\left(\frac{1}{2n-1} - \frac{1}{\sqrt{2n}}\right) = \frac{n}{2n-1} - \sqrt{\frac{n}{2}} < -1$, for $n \geq 5$. So $\frac{1}{2n-1} - \frac{1}{\sqrt{2n}} < -\frac{1}{n}$ for all but a finite number of terms: we have an infinite series of negative terms each less than the corresponding term of $-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right)$; so our series diverges to $-\infty$ by Cauchy's **Comparison Test**: *If $0 \leq a_n \leq b_n$ for all n, then if $b_1 + b_2 + b_3 + \ldots$ converges then so does $a_1 + a_2 + a_3 + \ldots$; if the latter diverges then so does the former.*

Leibniz formulated this test in 1682 and communicated it to Johann Bernoulli in letters dated 1713 and 1714. The idea of convergence was not established rigorously until Cauchy a hundred years later.

Web link: Proofs: divergence of the harmonic series web.williams.edu/Mathematics/lg5/exposit.html; convergence of alternating harmonic series www.maa.org/sites/default/files/Hudleson-MMz-201007804.pdf; divergence of non-descending alternating series example collegemathteaching.wordpress.com/2015/10/29/.

Further reading: *Calculus Deconstructed* by Zbigniew H. Nitecki, MAA, 2009, chapter 6.