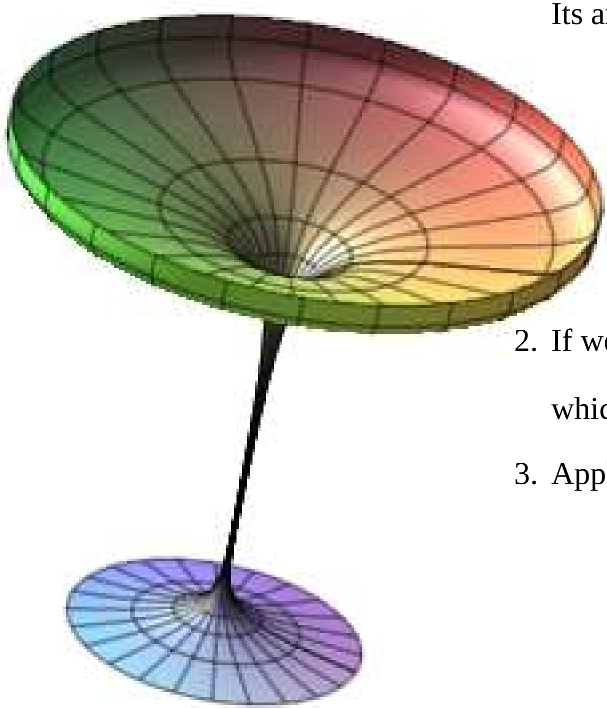
THEOREM OF THE DAY

The Basel Problem $1+\frac{1}{4}+\frac{1}{9}+\ldots=\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\tau^2}{24}$.

We write τ for one 'turn' (2π radians). A physical instance is the 'volume of revolution' as in the rotation of the curve $y = 1/\sqrt{2(1+x^6)}$ through τ radians, giving the champagne coupe shown on the right (the base has been added). If the stem is extended to become infinitely long then the volume of champagne contained in the glass is $\frac{\tau}{2}\int_0^{\infty}\frac{1}{2(1+x^6)}\,dx=\frac{\tau^2}{24}$. That this is also the sum of all reciprocals of squares is less easy to demonstrate. We give an informal summary of one proof on the right.



The Basel Problem, to express in closed form the value of $\sum 1/k^2$, was posed in the 1640s by Pietro Mengoli of Bologna, one of the early pioneers of the calculus. It defied many illustrious mathematicians over the next century before (1734) Euler made his name by solving it. Many beautiful proofs now exist; the one given here has been dubbed "Lewin's argument" (see web link below) but may well have been known to Euler who introduced the dilogarithm function and discovered equation (3) on the right.

1. Suppose we define the **dilogarithm** function $\mathrm{Li}_2(z)$ (pronounced 'lie 2 of z') thus:
$$\mathrm{Li}_2(z)=z+\frac{z^2}{4}+\frac{z^3}{9}+\frac{z^4}{16}+\ldots=\sum_{k=1}^{\infty}\frac{z^k}{k^2}. \qquad (1)$$
Its argument z is a complex number but our focus will be on just a couple of integers:
$$\mathrm{Li}_2(1)=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots,\quad \text{and}\quad \mathrm{Li}_2(-1)=-1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}-\ldots,$$
$$\text{whence}\quad \mathrm{Li}_2(1)+\mathrm{Li}_2(-1)=2\left(\frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\ldots\right)=2\times\frac{1}{4}\mathrm{Li}_2(1),$$
$$\text{so}\quad \mathrm{Li}_2(-1)=-\frac{1}{2}\mathrm{Li}_2(1). \qquad (2)$$

2. If we differentiate equation (1) and multiply by $-z$ we get $-z\mathrm{Li}_2'(z)=-z-\frac{z^2}{2}-\frac{z^3}{3}-\ldots$ which is the power series expansion of $\ln(1-z)$. So $\mathrm{Li}_2'(z)=-\frac{\ln(1-z)}{z}$.

3. Applying the chain rule,
$$\text{(a) } \mathrm{Li}_2'(-1/z) = -\frac{\ln(1-(-1/z))}{-1/z}\times\frac{1}{z^2} = \frac{1}{z}(\ln(1+z)-\ln z).$$
$$\text{(b) } \mathrm{Li}_2'(-z) = -\frac{\ln(1-(-z))}{-z}\times -1 = -\frac{1}{z}\ln(1+z).$$
$$\text{(c) } \frac{d}{dz}\frac{1}{2}(\ln(z))^2 = \frac{\ln z}{z}.$$

4. Now from (a), (b) and (c) we deduce that, for some constant C,
$$\mathrm{Li}_2(-1/z)+\mathrm{Li}_2(-z)+\frac{1}{2}(\ln(z))^2=C \qquad (3)$$
because the left-hand side differentiates to zero.

5. What is C? Put $z=1$ in equation (3): $\mathrm{Li}_2(-1)+\mathrm{Li}_2(-1)+0=C$ so, by equation (2), $C=2\mathrm{Li}_2(-1)=-\mathrm{Li}_2(1)$.

6. Now put $z=-1$ in equation (3), with $-\mathrm{Li}_2(1)$ replacing C:
$$\mathrm{Li}_2(1)+\mathrm{Li}_2(1)+\frac{1}{2}(\ln(-1))^2=-\mathrm{Li}_2(1),\quad \text{or}$$
$$\mathrm{Li}_2(1)=-\frac{1}{6}(\ln(-1))^2.$$
$$\text{But } -1=e^{i\tau/2},\quad \text{so}\quad \sum_{k=1}^{\infty}\frac{1}{k^2}=\mathrm{Li}_2(1)=-\frac{1}{6}\left(\ln\left(e^{i\tau/2}\right)\right)^2=-\frac{1}{6}\left(\frac{i\tau}{2}\right)^2=\frac{1}{24}\tau^2.$$

Web link: Dan Kalman and Mark McKinzie's award-winning article, 2013 at www.maa.org/programs/maa-awards/writing-awards/paul-halmos-lester-ford-awards. A 1975 winner was Raymond Ayoub's article reproduced in

Further reading: *The Genius of Euler* by William Dunham (ed.), MAA, 2007.