THEOREM OF THE DAY

Cardano’s Cubic Formula Given a cubic polynomial \( F(x) = x^3 + px + q \), let \( Z \) denote the square root \( \sqrt{(q/2)^2 + (p/3)^3} \). Let \( A \) and \( B \) be cube roots of \( -q/2 + Z \) and \( -q/2 - Z \), respectively, satisfying \( AB = -p/3 \). Then \( A + B \) is a solution of the equation \( F(x) = 0 \).

If the curve \( y = x^3 + ax^2 + bx + c \) is shifted to the right by \( a/3 \) via the substitution \( x := X - a/3 \) the effect is to put the equation into the form \( y = X^3 + pX + q \), for some \( p \) and \( q \). Setting \( y = 0 \) is now the same as asking for the points at which the curve \( y = X^3 \) meets the straight line \( y = -pX - q \). If \( p \) is positive, this straight line will slope negatively (top-left to bottom right) and there will be only one real-number solution. It took the genius of Cardano to realise that, when \( p \) was negative, the values of \( A + B \) in his theorem could become sums of conjugate complex numbers, giving three real-number solutions. For example, the cubic \( y = x^3 - 6x^2 - 9x + 14 \) becomes \( y = X^3 - 21X - 20 \) under the substitution \( x := X - (-6)/3 \). The cube roots of \( -q/2 + Z \) and \( -q/2 - Z \) (found via De Moivre’s Theorem) are plotted top-right as shown (the \( A_i \) and \( B_i \), respectively). The conjugate pairs \((A_0, B_0)\), \((A_1, B_2)\) and \((A_2, B_1)\) satisfy \( A_iB_j = -p/3 \) and add to give real roots 5, -1 and -4 of \( X^3 - 21X - 20 \) (and we shift left by -2 to get roots 7, 1 and -2 for the original equation). By contrast, for the equation \( y = X^3 + 14X + 300 \), the roots \( A_1 \) and \( B_1 \) (bottom-right) do not combine in conjugate pairs. Although three pairs satisfy \( A_iB_j = -p/3 \) only \( A_0 \approx 0.6968 \) and \( B_0 = -6 - A_0 \) sum to give a real root of \( X = -6 \).

Web link: capone.mtsu.edu/jhart/cardon.pdf

Gerolamo Cardano (1501–1576) has been accused of stealing from Nicolo Tartaglia (1500–1557) the solution of the cubic. However, a solution had already been published by Scipione del Ferro (1465–1526). Both solutions were acknowledged by Cardano who moreover surpassed them with the above formula which alone addressed the 3-real-number case. And it was perhaps Rafael Bombelli (1525–1572) who first really understood the role played by complex numbers. This was well over 100 years before De Moivre’s theorem: complex-based solutions could be constructed but the cubic equation was still not solvable explicitly, in general.