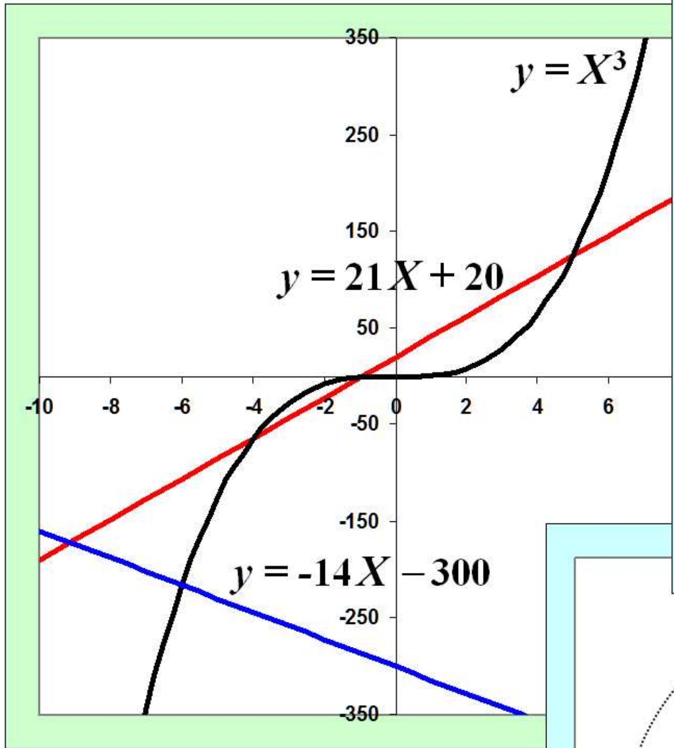
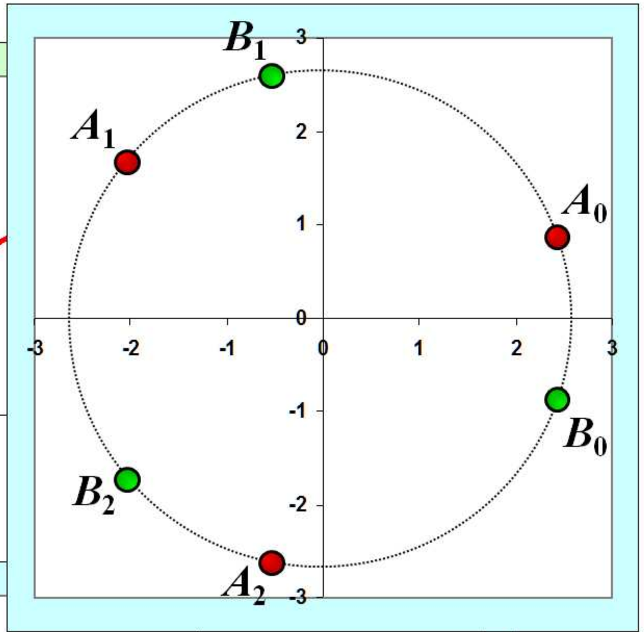
THEOREM OF THE DAY

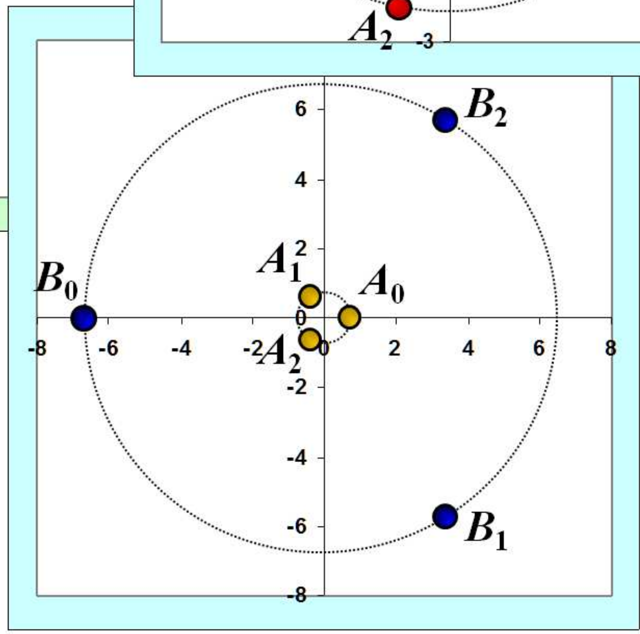
Cardano's Cubic Formula *Given a cubic polynomial $F(x) = x^3 + px + q$, let Z denote the square root $\sqrt{(q/2)^2 + (p/3)^3}$. Let A and B be cube roots of $-q/2+Z$ and $-q/2-Z$, respectively, satisfying $AB = -p/3$. Then $A + B$ is a solution of the equation $F(x) = 0$.*





$$A_0 = \frac{5}{2} + i\frac{\sqrt{3}}{2}$$
$$A_1 = -2 + i\sqrt{3}$$
$$A_2 = -1/2 - i\frac{3\sqrt{3}}{2}$$

$$B_0 = \frac{5}{2} - i\frac{\sqrt{3}}{2}$$
$$B_1 = -\frac{1}{2} + i\frac{3\sqrt{3}}{2}$$
$$B_2 = -2 - i\sqrt{3}$$



If the curve $y = x^3 + ax^2 + bx + c$ is shifted to the right by $a/3$ via the substitution $x := X - a/3$ the effect is to put the equation into the form $y = X^3 + pX + q$, for some p and q. Setting $y = 0$ is now the same as asking for the points at which the curve $y = X^3$ meets the straight line $y = -pX - q$. If p is positive, this straight line will slope *negatively* (top-left to bottom right) and there will be only one real-number solution. It took the genius of Cardano to realise that, when p was negative, the values of $A + B$ in his theorem could become sums of conjugate complex numbers, giving three real-number solutions. For example, the cubic $y = x^3 - 6x^2 - 9x + 14$ becomes $y = X^3 - 21X - 20$ under the substitution $x := X - (-6)/3$. The cube roots of $-q/2 + Z$ and $-q/2 - Z$ (found via De Moivre's Theorem) are plotted top-right as shown (the A_i and B_i, respectively). The conjugate pairs (A_0, B_0), (A_1, B_2) and (A_2, B_1) satisfy $A_iB_j = -p/3$ and add to give real roots 5, -1 and -4 of $X^3 - 21X - 20$ (and we shift left by -2 to get roots 7, 1 and -2 for the original equation). By contrast, for the equation $y = X^3 + 14X + 300$, the roots A_i and B_i (bottom-right) do not combine in conjugate pairs. Although three pairs satisfy $A_iB_j = -p/3$ only $A_0 \approx 0.6968$ and $B_0 = -6 - A_0$ sum to give a real root of $X = -6$.

Gerolamo Cardano (1501–1576) has been accused of stealing from Nicolo Tartaglia (1500–1557) the solution of the cubic. However, a solution had already been published by Scipione del Ferro (1465–1526). Both solutions were acknowledged by Cardano who moreover surpassed them with the above formula which alone addressed the 3-real-number case. And it was perhaps Rafael Bombelli (1525–1572) who first really understood the role played by complex numbers. This was well over 100 years before De Moivre's theorem: complex-based solutions could be constructed but the cubic equation was still not solvable explicitly, in general.

Web link: capone.mtsu.edu/jhart/cardan.pdf

Further reading: *Why Beauty is Truth: the History of Symmetry* by Ian Stewart, Basic Books, 2008, chapter 4.