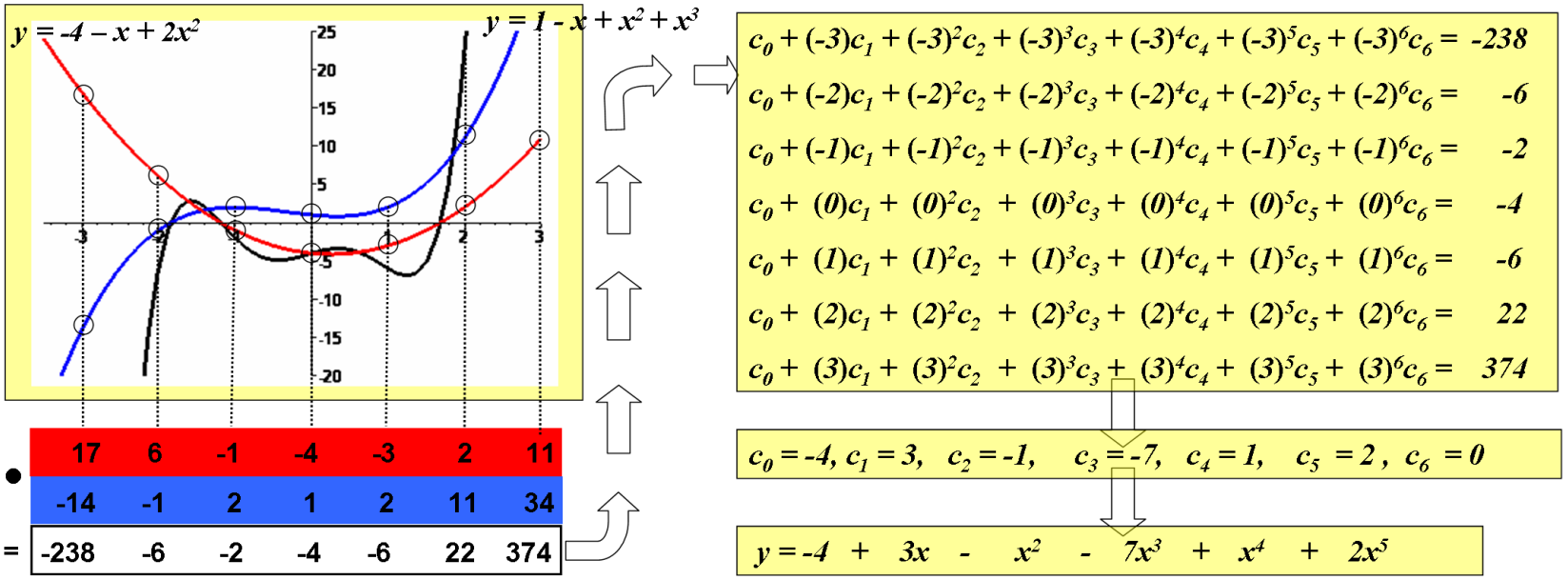
THEOREM OF THE DAY

The Convolution Theorem *Let* $a = (a_0, \ldots, a_n)$ *and* $b = (b_0, \ldots, b_n)$ *be vectors in* $\mathbb{C}^{n+1}$. *The* convolution *of* a *and* b, *denoted* $a \star b$, *is the vector* $c = (c_0, \ldots, c_{2n})$, *in* $\mathbb{C}^{2n+1}$, *defined by* $c_i = \sum_{j=0}^{n} a_j b_{i-j}$, $i = 0, \ldots, 2n$, *with* $b_k = 0$ *whenever* $k < 0$ *or* $k > n$. *Then*

$$a \star b = \mathcal{F}^{-1}(\mathcal{F}(a) \bullet \mathcal{F}(b)),$$

where $\mathcal{F}$ *is the* $(2n+1)$*-dimensional Discrete Fourier Transform and* $\bullet$ *is componentwise multiplication.*



The picture shows a version of the Convolution Theorem for polynomials: $a \star b$ = interpolate(evaluate(a) $\bullet$ evaluate(b)). 'Evaluate' means evaluate a degree n polynomial at $2n + 1$ points; the Discrete Fourier Transform (DFT) corresponds to a particular choice of points. Here we have merely chosen $2n + 1 = 7$ integers at which the two polynomials a : $-4 - x + 2x^2$ and b : $1 - x + x^2 + x^3$ are evaluated (polynomial a is taken to have degree 3 but with $a_3 = 0$). These values are multiplied and interpolation is applied: the 7 component products define a system of 7 simultaneous equations whose unique solution is precisely the list of coefficients of the product of the two polynomials (the coefficient of x^6 is zero because polynomial a had degree < 3). For polynomials, this is precisely what convolution achieves.

The DFT evaluates instead at complex numbers: the $2n + 1$ powers of a primitive $(2n + 1)$-th root of unity. Via Euler's Identity, these are expressed as powers of $e^{\tau i/(2n+1)}$ ($\tau = 2\pi$). In general, to apply the DFT to a vector v of length N, we multiply v by the $N \times N$ matrix $\mathcal{F}$ whose ij-th element is $\omega^{(i-1)(j-1) \pmod N}$, $\omega = e^{\tau i/N}$. (To our polynomial coefficient vectors we first need to append n zeros to extend to length $2n + 1$.) This special choice of points leads to a dramatic computational short-cut: the so-called *Fast* Fourier Transform (FFT) achieves convolution (and hence polynomial multiplication) in time $O(N \log N)$ instead of $O(N^2)$.

This discrete convolution theorem is intimately connected with the FFT known, in some form, to Gauss, as early as 1805; rediscovered by Cornelius Lanczos in 1940; and made widely known by James Cooley and John Tukey, 1965.

Web link: betterexplained.com/articles/intuitive-convolution/

Further reading: *The Design and Analysis of Computer Algorithms* by A. Aho, J.E. Hopcroft and J.Ullman, Addison-Wesley, 1974, chapter 7.