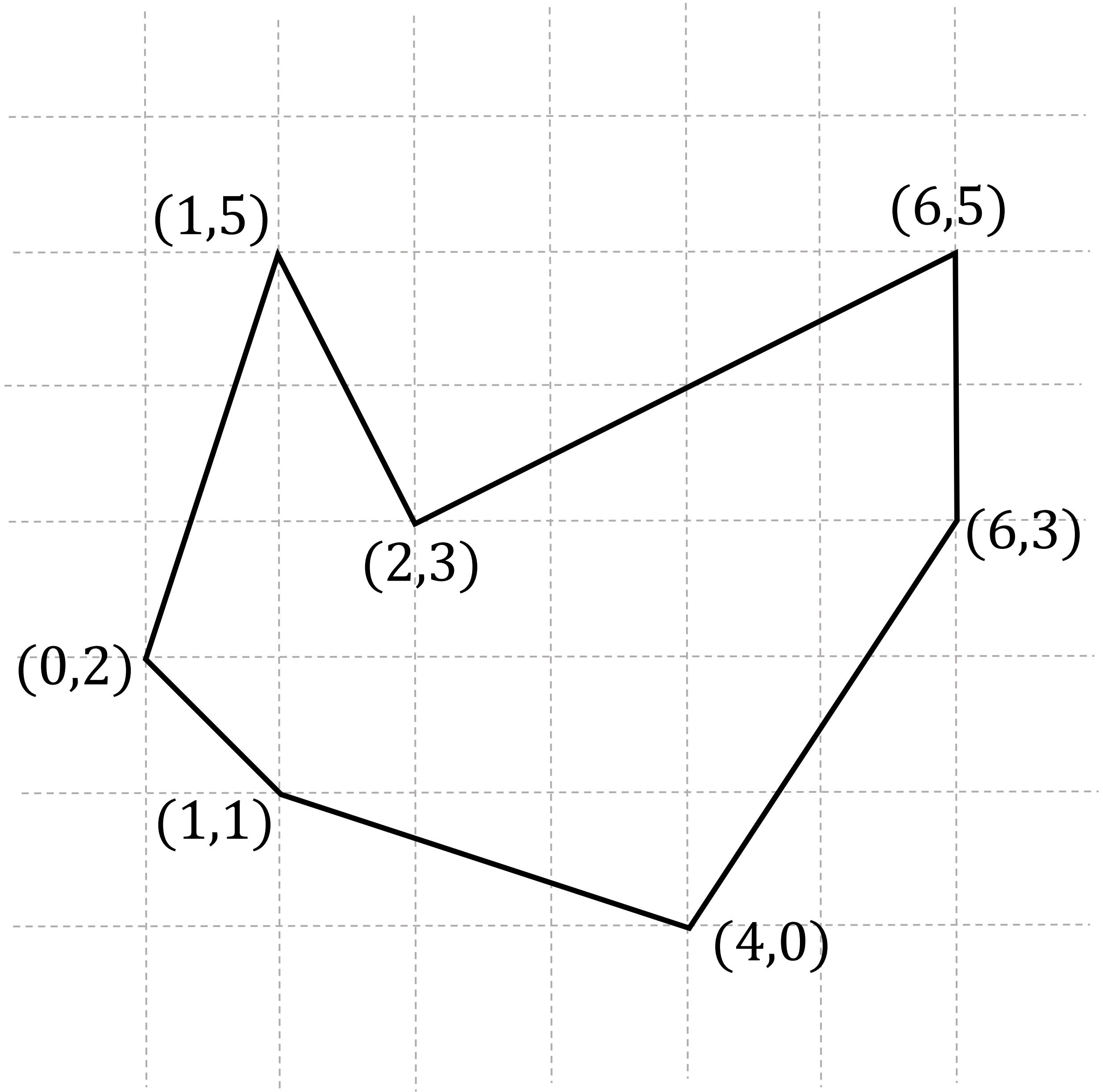
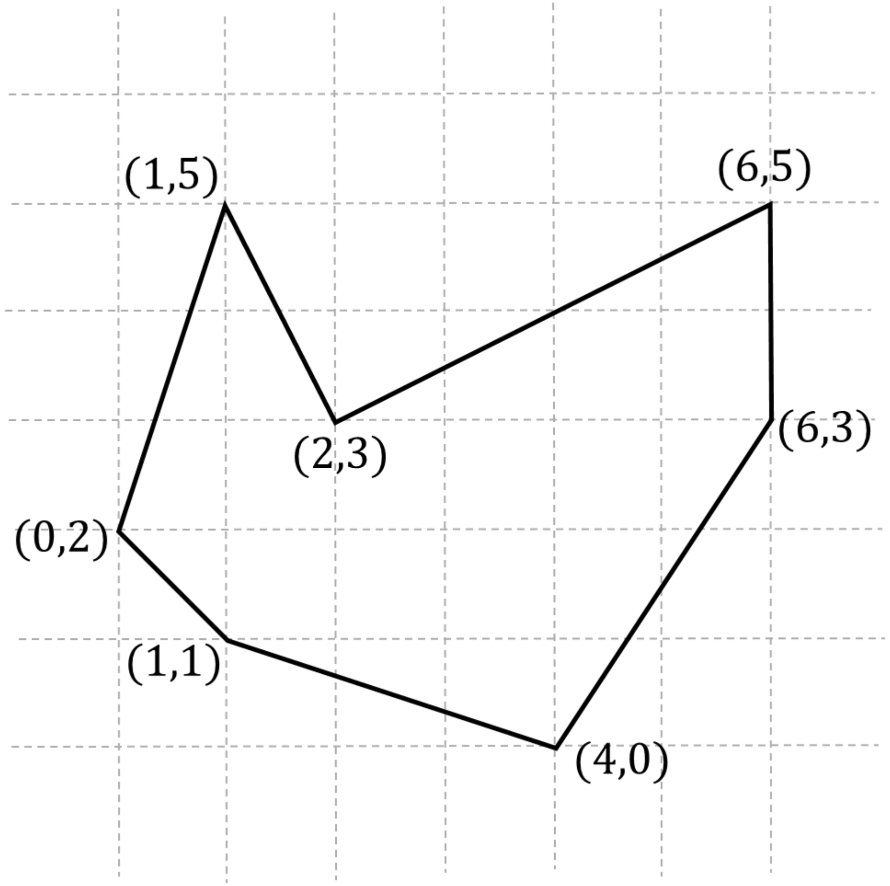
How convex is this polygon?

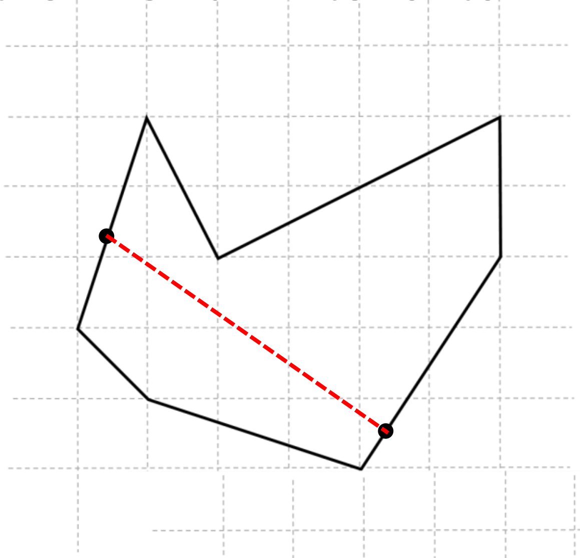


Is this polygon convex?

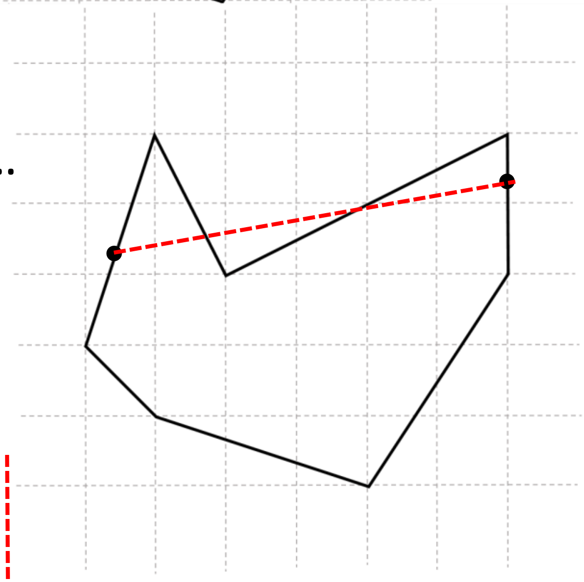


For any two points on the boundary, does the straight line joining them remain interior to the polygon?

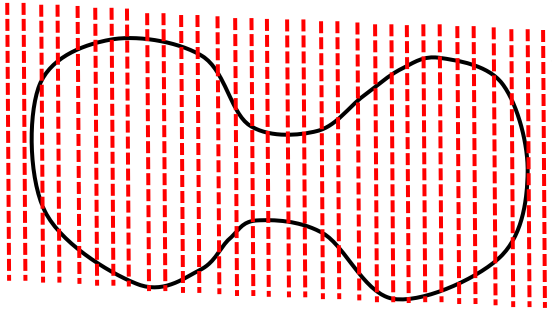
Maybe…



Or maybe not…

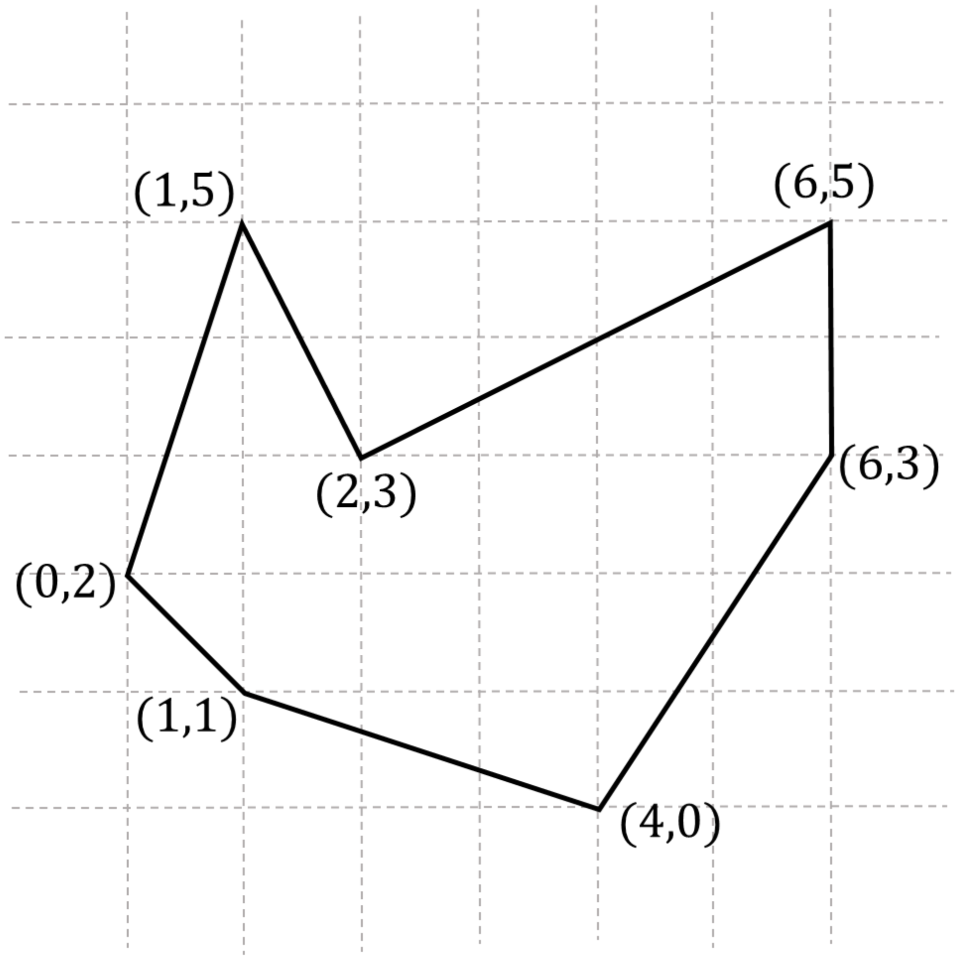


How can we systematically check every pair of points?
A **certificate of non-convexity** is easy to describe – what about a certificate of convexity?
Is this even **algorithmic**?



Is this a **certificate of convexity?**

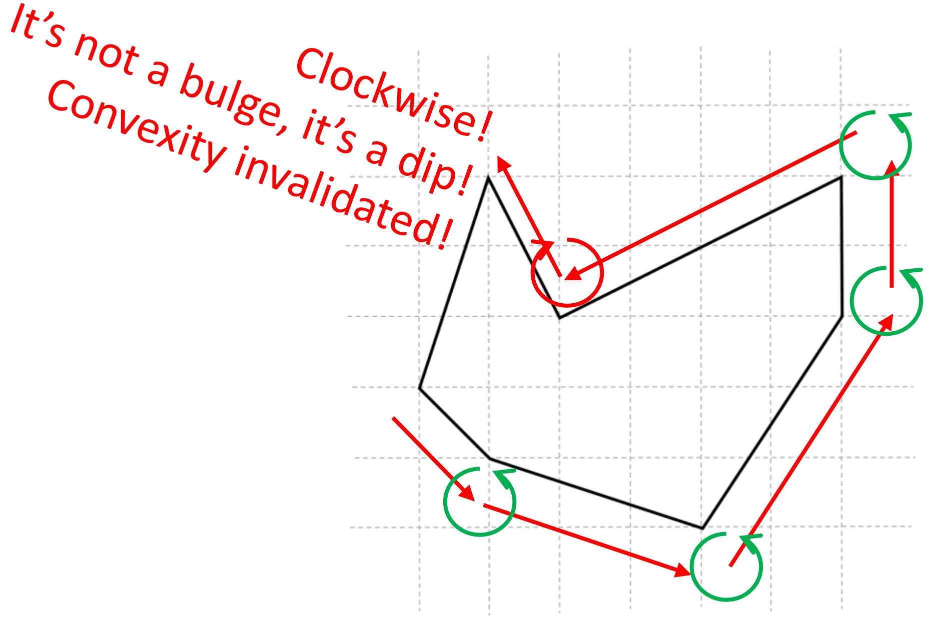
Suppose our polygon is given as a set of consecutive edge vectors



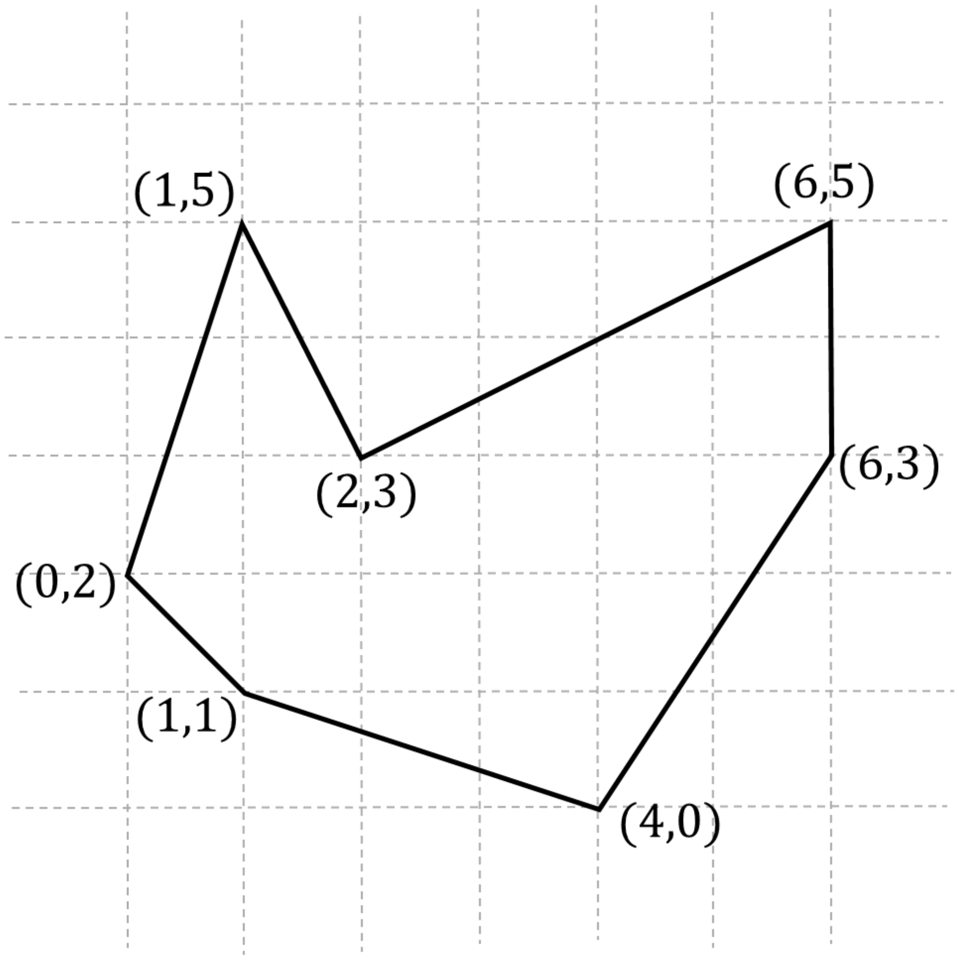
Or, same thing, a list of coordinates of vertices.

The edge from (a, b) to (c, d) is the direction vector $-(a, b) + (c, d)$

We will follow consecutive direction vectors and make sure they always turn in the same direction



Cross product and direction of turn



Suppose $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ and $d\boldsymbol{i} + e\boldsymbol{j} + f\boldsymbol{k}$ are two vectors in three dimensions. The cross product

$$a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} \times d\boldsymbol{i} + e\boldsymbol{j} + f\boldsymbol{k}$$

is a vector in the direction perpendicular to their common plane. It is conveniently calculated as a determinant:

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a & b & c \\ d & e & f \end{vmatrix}$$

It is positive if our two vectors follow each other counterclockwise and negative otherwise.

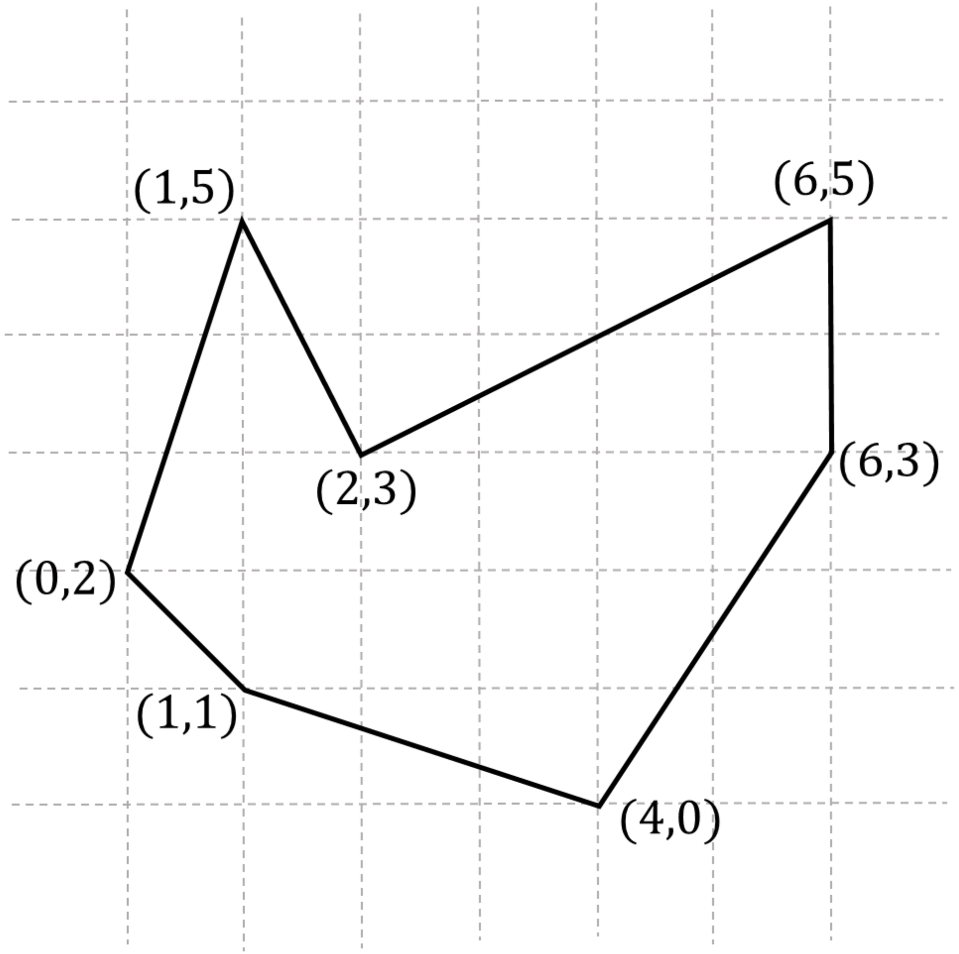
If we are in the plane then $c = f = 0$ and we just have

$$a\boldsymbol{i} + b\boldsymbol{j} \times d\boldsymbol{i} + e\boldsymbol{j} = (ae - bd)\,\boldsymbol{k}$$

and we may ignore the fact that this is a vector.

If the 2×2 determinant is positive we are turning counterclockwise, otherwise we are turning clockwise.

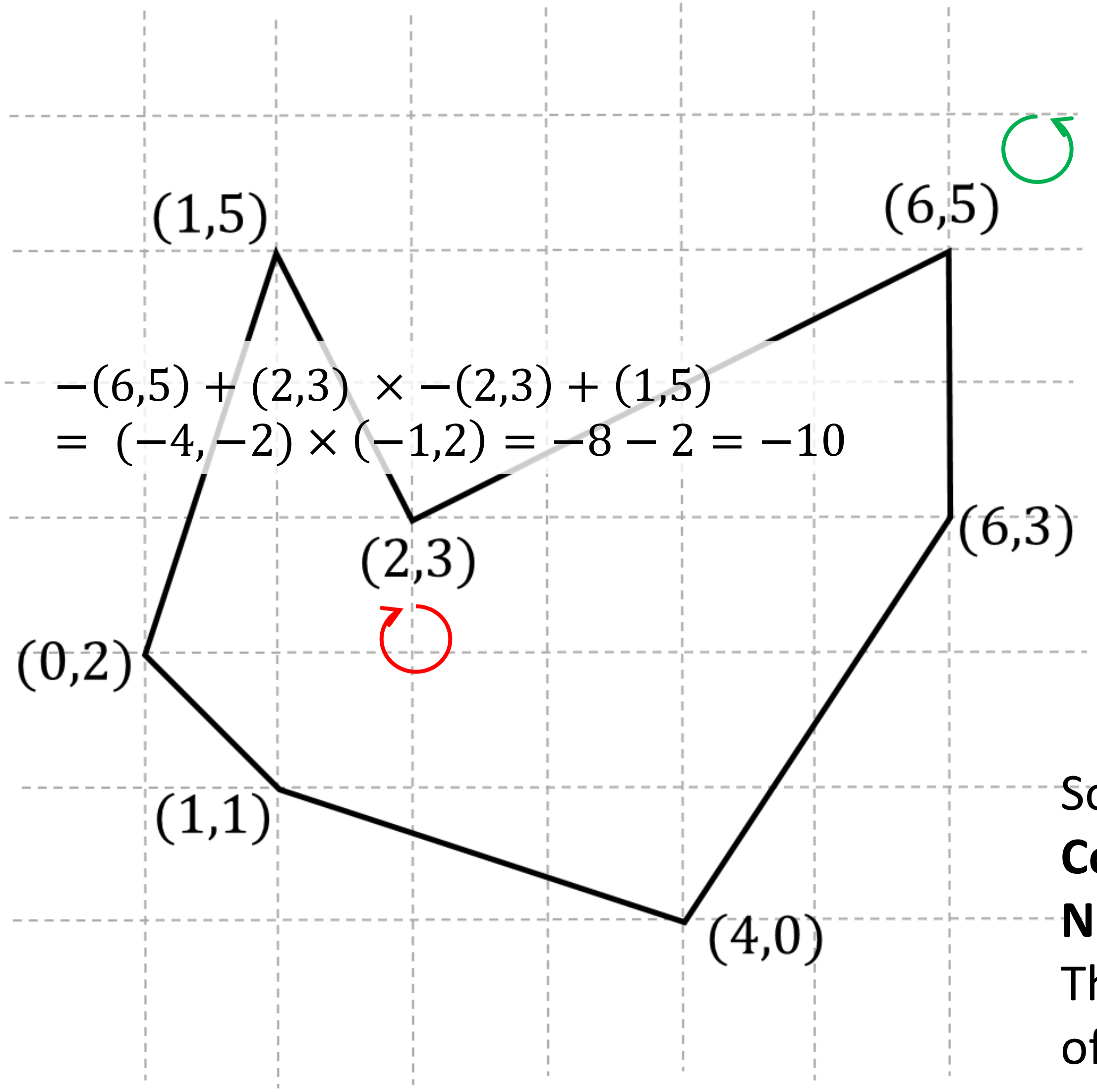
Compare to Shoelace formula



If the n vertices of a polygon are specified as position vectors $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \ \boldsymbol{v}_{n-1}$, then the area of the polygon is half the sum of the cross poducts: $\boldsymbol{v}_i \times \boldsymbol{v}_{i+1}, i = 0, \ldots, n-1$.

$$2 \times \text{Area} = (0,2) \times (1,1) + (1,1) \times (4,0) + \cdots + (1,5) \times (0,2).$$

Cross product and direction of turn in the plane



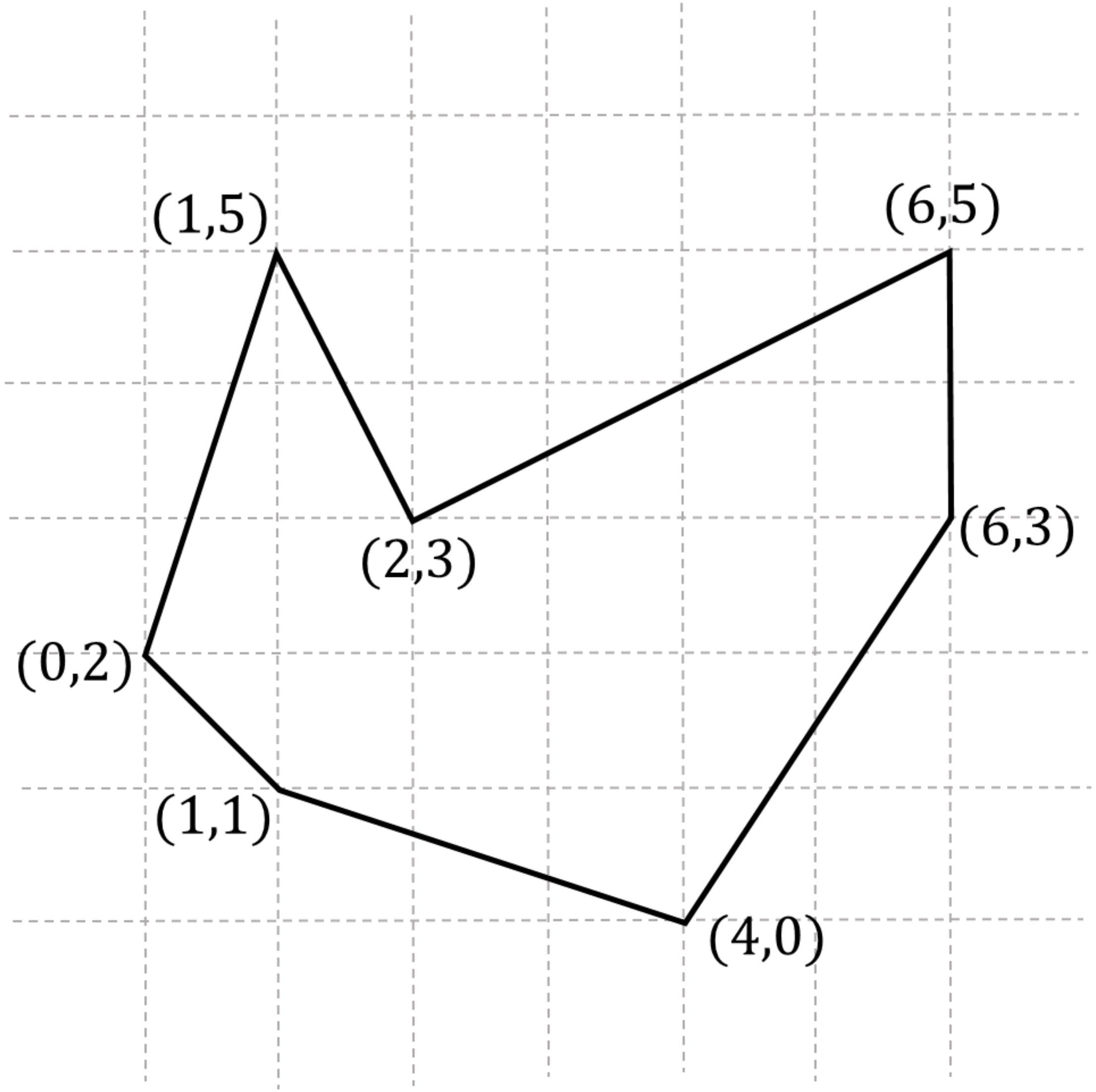
$$-(6,5) + (2,3) \times -(2,3) + (1,5)$$
$$= (-4,-2) \times (-1,2) = -8 - 2 = -10$$

$$-(6,3) + (6,5) \times -(6,5) + (2,3)$$
$$= (0,2) \times (-4,-2) = 0 - -8 = 8$$

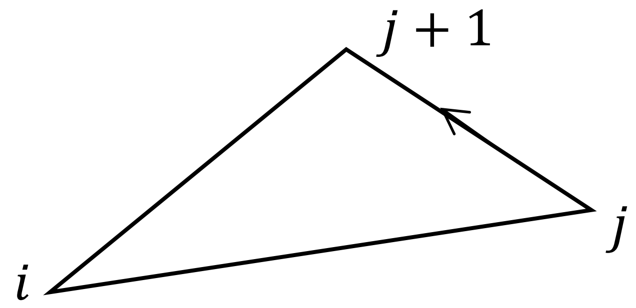
$$-(4,0) + (6,3) \times -(6,3) + (6,5)$$
$$= (2,3) \times (0,2) = 4 - 0 = 4$$

So we have our certificates:
Convex: list of n non-negative cross products
Non-convex: single negative cross product
This is even a linear test (in the input size, say, list of vertex coordinates)

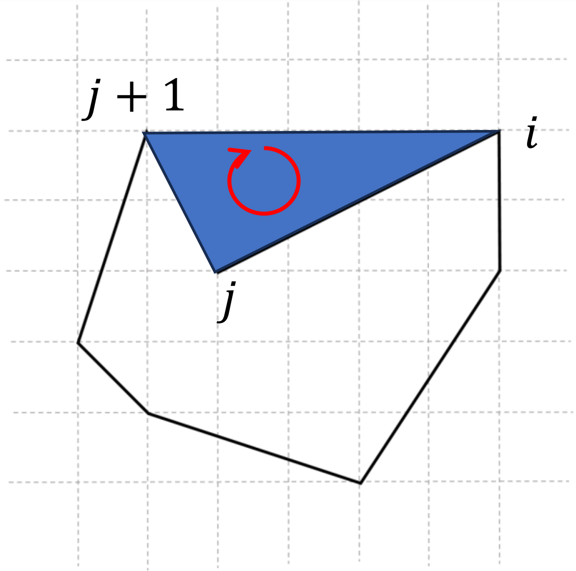
Convexity by triangulation



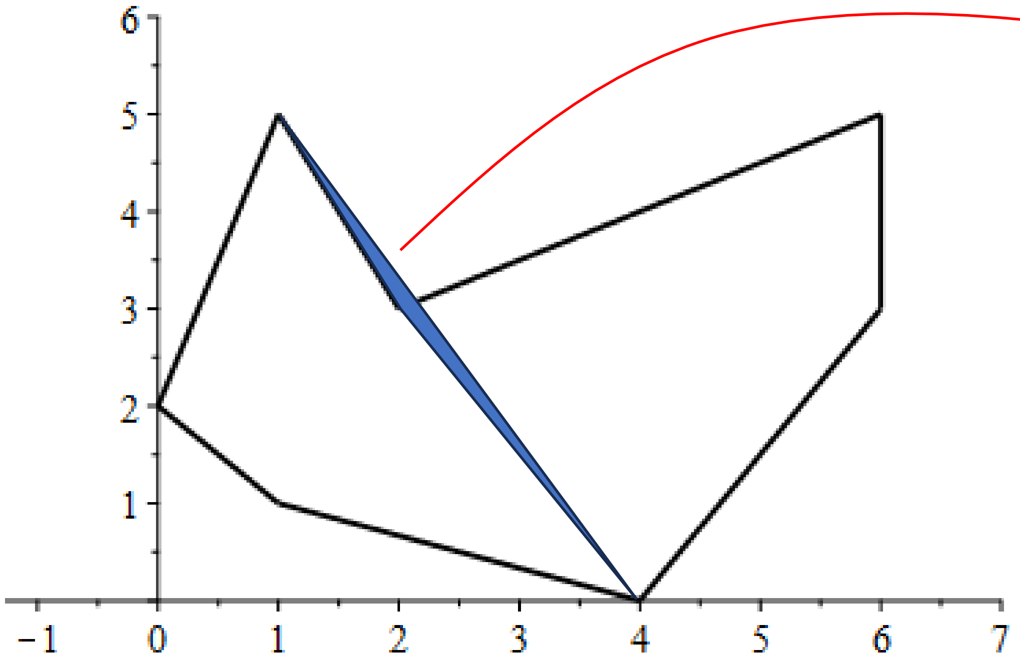
From vertex i we take the areas of all triangles subtended on opposite edges



The area is positive if the triangle has a counterclockwise orientation, otherwise it is negative



The matrix $\Delta_{i,j}$



$$sm := \begin{bmatrix} 0 & 1 & 8 & 6 & 0 & \frac{5}{2} & 0 \\ 0 & 0 & \frac{11}{2} & 5 & 3 & 2 & 2 \\ 1 & 0 & 0 & 2 & 8 & -\frac{1}{2} & 7 \\ \frac{7}{2} & \frac{11}{2} & 0 & 0 & 4 & -4 & \frac{17}{2} \\ \frac{9}{2} & \frac{17}{2} & 2 & 0 & 0 & -5 & \frac{15}{2} \\ \frac{3}{2} & \frac{7}{2} & 6 & 4 & 0 & 0 & \frac{5}{2} \\ 2 & 6 & \frac{19}{2} & 5 & -5 & 0 & 0 \end{bmatrix}$$

So this is a matrix full of certificates. But it’s quadratic for convexity testing isn’t it?
Surprisingly not.

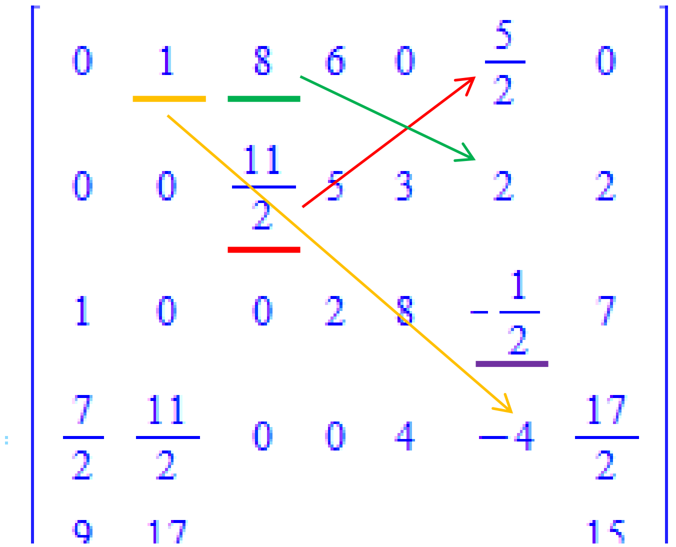
$\Delta_{i,j}$ has rank 3

$$\begin{bmatrix} 0 & 1 & 8 & 6 & 0 & \frac{5}{2} & 0 \\ 0 & 0 & \frac{11}{2} & 5 & 3 & 2 & 2 \\ 1 & 0 & 0 & 2 & 8 & -\frac{1}{2} & 7 \\ \frac{7}{2} & \frac{11}{2} & 0 & 0 & 4 & -4 & \frac{17}{2} \\ \frac{9}{2} & \frac{17}{2} & 2 & 0 & 0 & -5 & \frac{15}{2} \\ \frac{3}{2} & \frac{7}{2} & 6 & 4 & 0 & 0 & \frac{5}{2} \\ 2 & 6 & \frac{19}{2} & 5 & -5 & 0 & 0 \end{bmatrix}$$

All this determined by first three rows

Depends on a rather unexpected relationship between products of triangle areas:

$$\Delta_{1,2}\Delta_{0,k} - \Delta_{0,2}\Delta_{1,k} - \Delta_{0,1}\Delta_{3,k} = (\Delta_{1,2} - \Delta_{0,2} - \Delta_{0,1})\,\Delta_{2,k}$$



$$\frac{11}{2} \times \frac{5}{2} - 8 \times 2 - 1 \times -4 = \left(\frac{11}{2} - 8 - 1\right) \times -\frac{1}{2}$$

A theorem about plane triangles

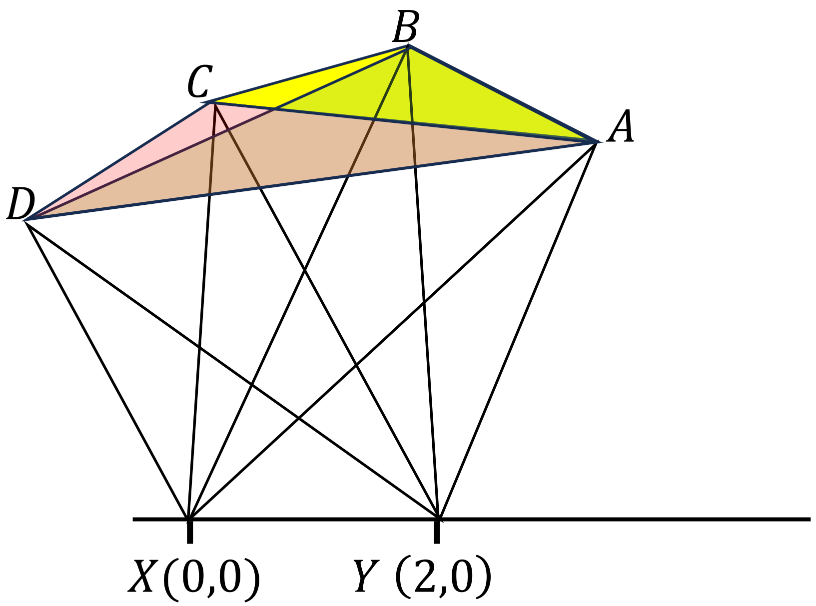
Let A, B, C, D, X, Y be six points, ordered counterclockwise, in the plane.
Let ABC, ABD, ACD, BCD, be the four triangles formed on points A, B, C, D, with areas $|ABC|$ etc.
Let $\Delta_A, \Delta_B, \Delta_C, \Delta_D$, be the four triangle areas formed by joining edge XY to points A, B, C, D, respectively.
Then

$$|ABC|\Delta_D - |ABD|\Delta_C + |ACD|\Delta_B - |BCD|\Delta_A = 0.$$

Without loss of generality, let X, Y be the points $(0,0)$ and $(2,0)$.

Now the areas Δ_A, etc are just the vertical coordinates of A, B, C, D, respectively.

The identity can be confirmed using the Shoelace formula.



Bisection envelopes (polygons)

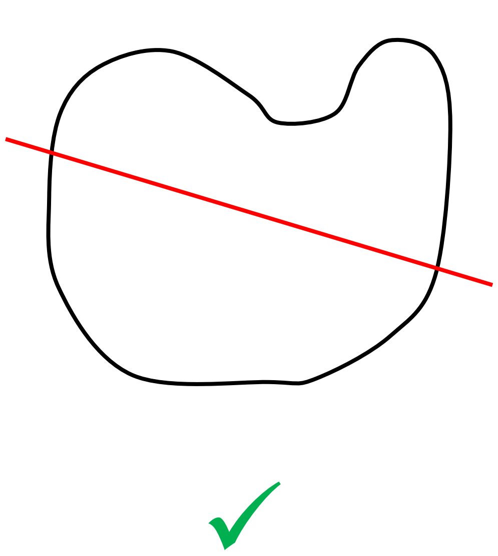
involve
a journal of mathematics

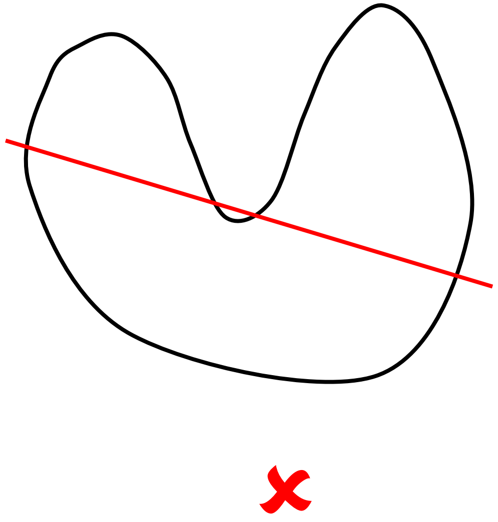
Bisection envelopes

Noah Fechtor-Pradines

Involve, Vol. 8 (2015) 307–328

Bisection-convex: any bisecting straight line intersects the curve in exactly two points





Strictly bisection-convex curves

We now restrict the class of curves $\mathcal{S}$ to be studied.

Definition 2.2. Define $\mathcal{S}$ and $\mathcal{L}$ as above. We say that $\mathcal{S}$ is *bisection convex* if for all θ, l_θ intersects $\mathcal{S}$ in exactly two points. Alternatively, for every point A on $\mathcal{S}$, there exists a unique point B also on $\mathcal{S}$ such that the line AB bisects the interior area of $\mathcal{S}$.

We also create a tighter restriction.

Definition 2.3. Define $\mathcal{S}$ and $\mathcal{L}$ as before. We say that $\mathcal{S}$ is *strictly bisection convex* if it is bisection convex and for all θ, l_θ is not tangent to $\mathcal{S}$. At any point where there are two tangents to $\mathcal{S}$ — one from each side — the l_θ through that point is distinct from both tangents.

Henceforth, unless otherwise stated, *it is assumed that $\mathcal{S}$ is strictly bisection convex.*

Bisection envelopes
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That IVT 2-pancakes issue again…

Define $A(\theta)$ and $B(\theta)$ to be the endpoints of the bisecting chord in direction θ, with $B(\theta) = A(\theta + \pi)$. We distinguish between $A(\theta)$ and $B(\theta)$ by demanding that for each point $Q \neq A(\theta), B(\theta)$ on the bisecting chord, the vector $A(\theta) - Q$ points in positive direction θ and the vector $B(\theta) - Q$ points in positive direction $\theta + \pi$.

Proposition 2.4. *Assume that S is bisection convex. Then $A(\theta)$ varies continuously with θ.*

Proof. First, we note that any two bisecting chords must intersect in the interior of S, for if they did not, the interior of S would be split into three regions, one of which would have zero area, which does not make sense.

From this, we have $\lim_{\epsilon \to 0} l_{\theta+\epsilon} = l_\theta$, as the limit of the intersection point $l_{\theta+\epsilon} \cap l_\theta$ is bounded. This also implies that the limit as $\epsilon \to 0$ of the distance from $A(\theta + \epsilon)$ to the intersection point $l_{\theta+\epsilon} \cap l_\theta$ is bounded. Therefore, the limit as $\epsilon \to 0$ of the perpendicular distance from $A(\theta + \epsilon)$ to l_θ is zero.

We have that $\lim_{\epsilon \to 0} A(\theta + \epsilon)$ must be a point P on l_θ which intersects S, where for every other point Q on the bisecting chord with direction θ, the vector $P - Q$ points in positive direction θ. There is only one such point, $A(\theta)$; therefore,

$$\lim_{\epsilon \to 0} A(\theta + \epsilon) = A(\theta),$$

and $A(\theta)$ varies continuously with θ. $\square$

Bisection envelopes

Noah Fechtor-Pradines

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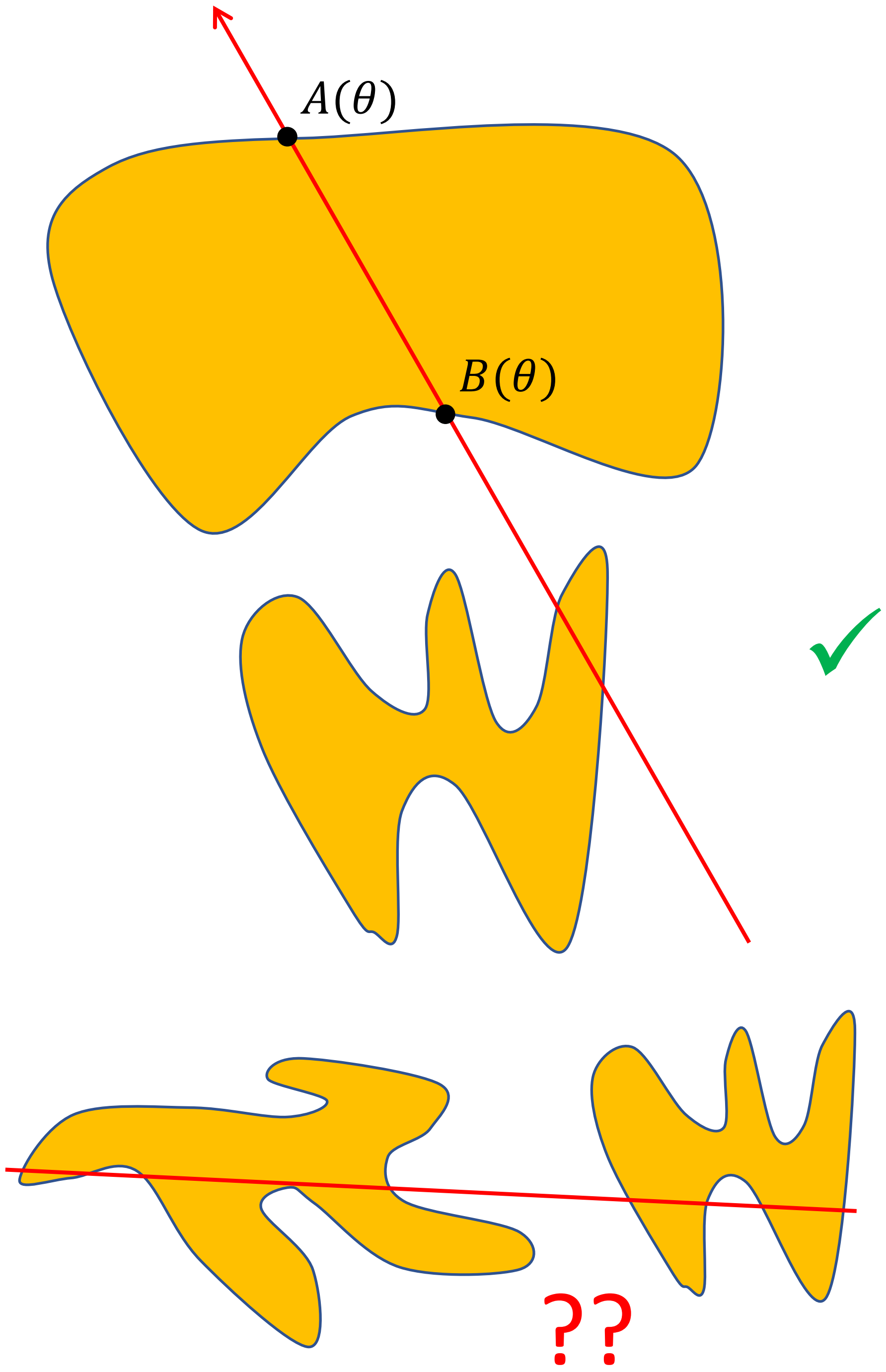
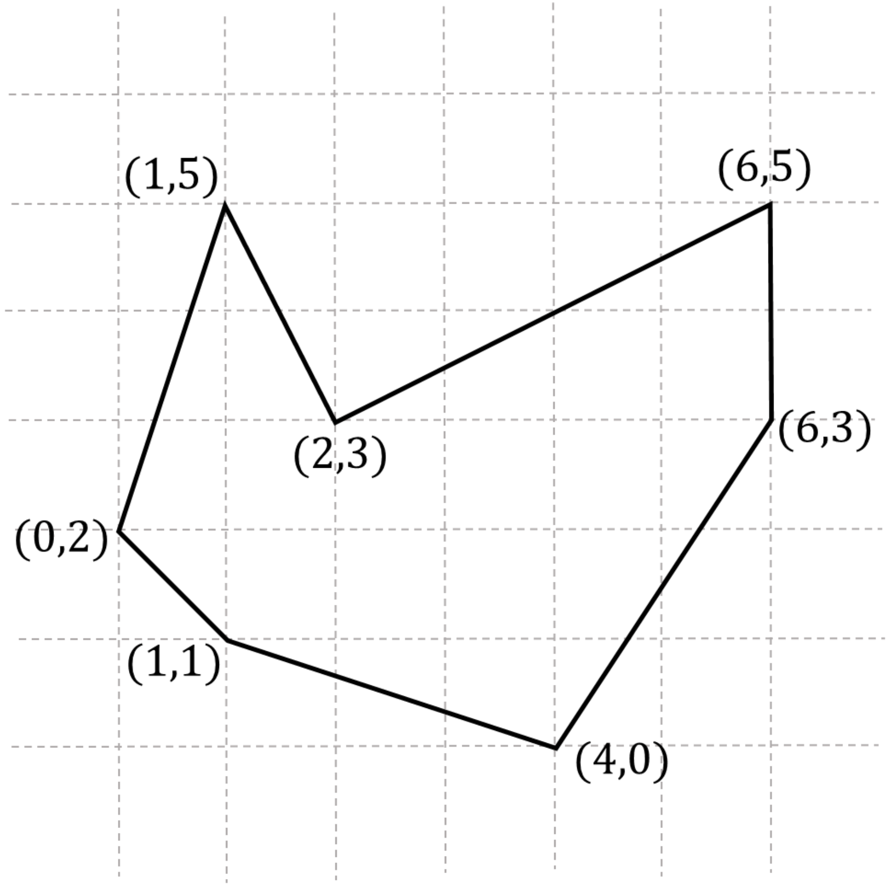
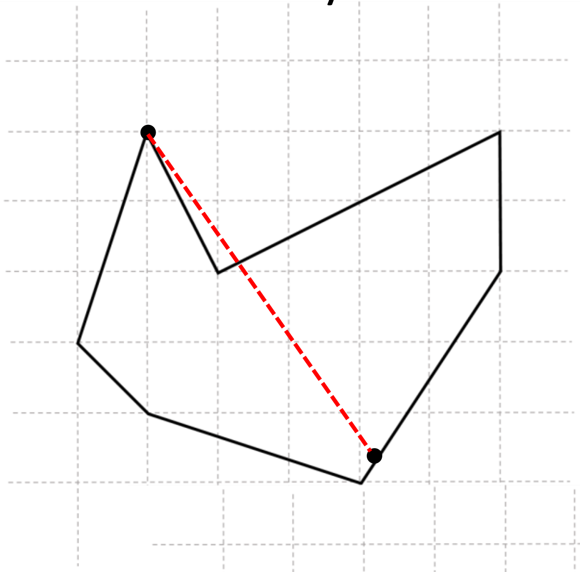


Is this polygon bisection-convex?

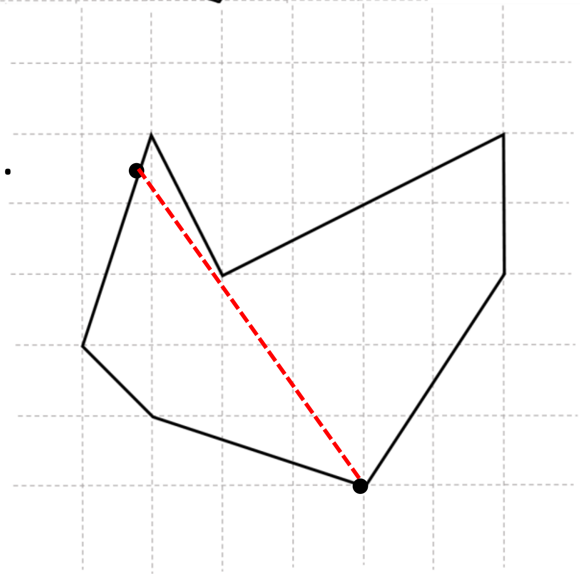
For any straight line bisecting the polygon, does it intersect the boundary in more than two points



Maybe…



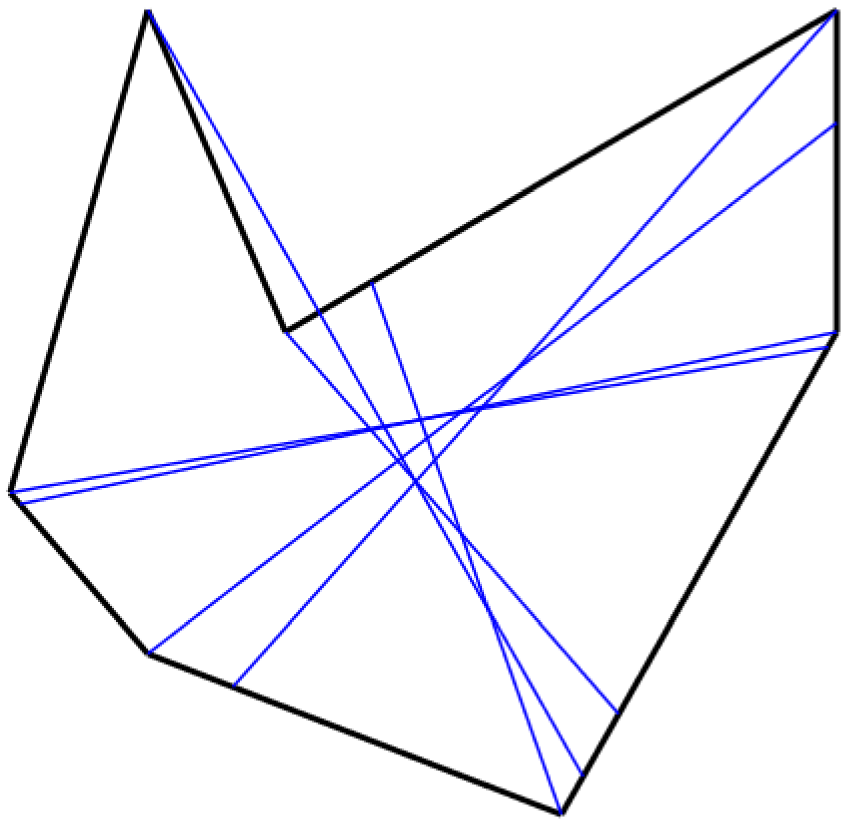
Or maybe not…



How can we systematically check every bisecting line?
A **certificate of non-bisection-convexity** is easy to describe – what about a certificate of bisection-convexity?
Is this even **algorithmic**?

A characterisation

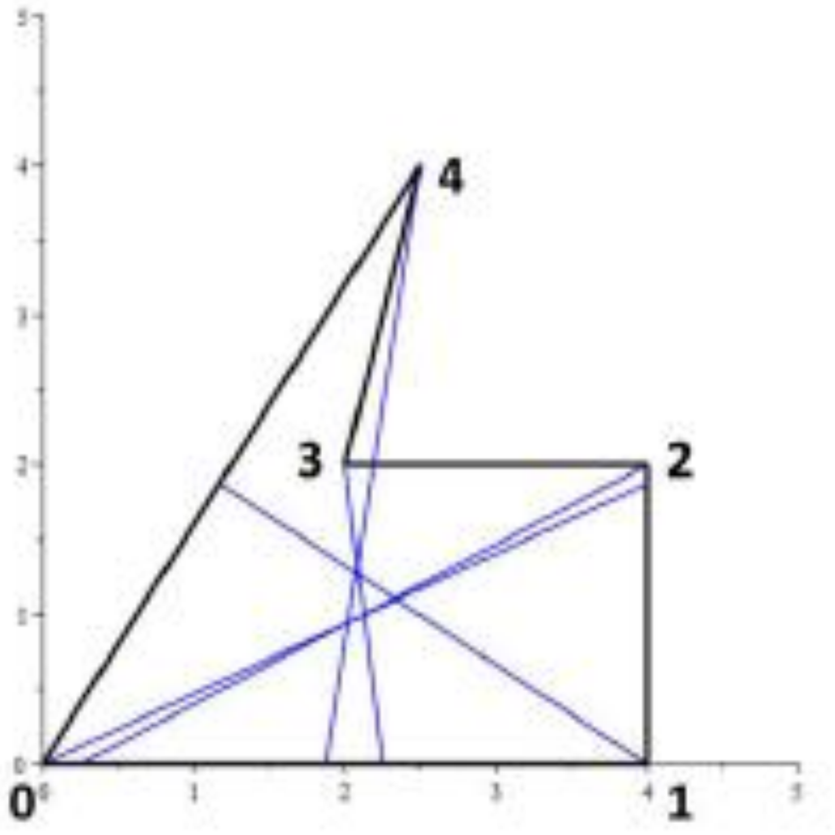
Let P be a polygon. For each vertex v of P let the unique straight line through v bisecting P be given as $\boldsymbol{r}_v$. Then P is bisection-convex if and only if no $\boldsymbol{r}_v$ intersects the boundary of P in three or more points.



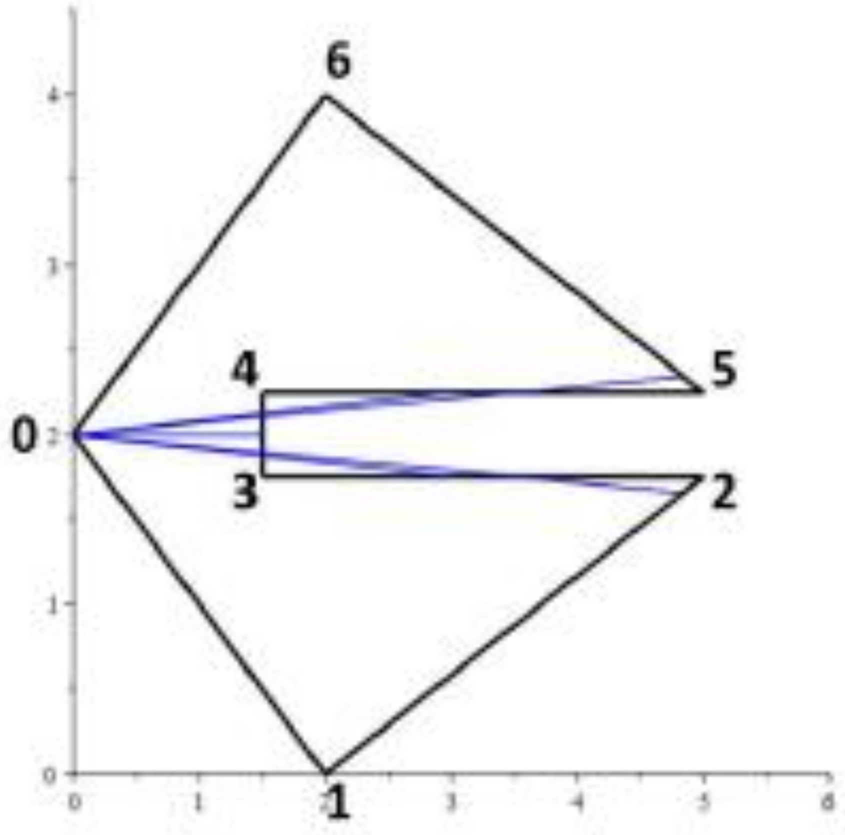
A **certificate of bisection-convexity** is a collection of n bisecting lines $\boldsymbol{r}_v$ which all lie within the boundary of P.

However, requires an effective test for a line to bisect P.

What is ‘bisecting’?



(a)

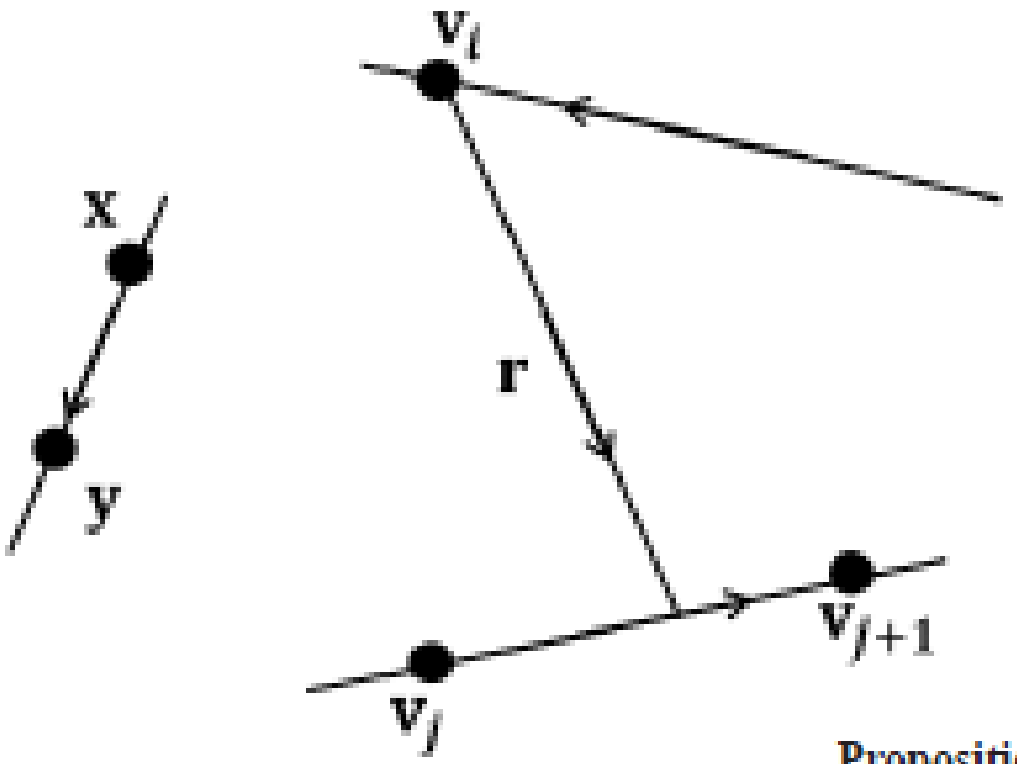


(b)

The line $\boldsymbol{r}_3$ bisects by joining vertex 3 to an opposite edge. However, this line extends to meet the polygon boundary elsewhere.

All five lines from vertex 0 ‘bisect’, in the sense that the two ‘half’ polygons joining the end-points of the lines both compute (Shoelace formula) half the area.

Is bisecting vector r crossed by edge xy?



Once again the cross product is the needed resource.

If vector r from vertex $\boldsymbol{v}_i$ crosses edge xy then

$$(-\boldsymbol{v}_i + \mathbf{x}) \times \boldsymbol{r} \text{ and } (-\boldsymbol{v}_i + \mathbf{y}) \times \boldsymbol{r}$$

will have different signs.

Luckily the sequence of cross products for the polygon edges lying counterclockwise from $\boldsymbol{v}_i$ may all be calculated from the edges of the triangle area matrix $\Delta_{i,j}$

Proposition 9 *Define the sequence F_i, $i \geq 0$, by*

$$F_0 = 0 \text{ and, for } k \geq 1, F_k = F_{k-1} + \Delta_{j,i+k-1} - \Delta_{i,j+k-1}.$$

Then for $k = 1, \ldots, n-1$,

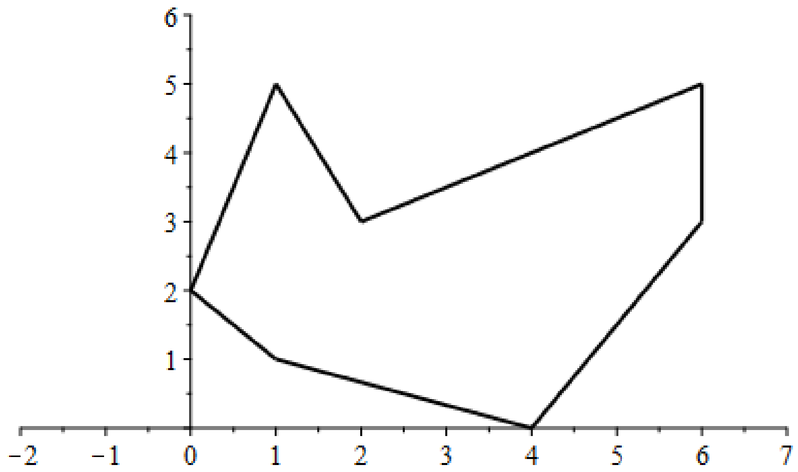
$$(-\mathbf{v}_i + \mathbf{v}_{i+k}) \times \mathbf{r} = 2r_{i,j}\left(\Delta_{i,j} - \Delta_{i+k,j}\right) + 2F_k.$$

What more can we say about $\Delta_{i,j}$?

The matrix $\Delta_{i,j}$ has rank 3, therefore $n-3$ zero eigenvalues. It is easy to calculate the corresponding eigenvectors using the plane triangle theorem.

The rows of $\Delta_{i,j}$ all sum to the area of the polygon, because the i-th row partitions the polygon into triangles subtended from vertex i. The area is therefore also an eigenvalue of $\Delta_{i,j}$. The corresponding eigenvector is the all-ones vector.

There remain two eigenvalues which are the (complex) roots of a quadratic polynomial. These are mysterious to me.



$$sm := \begin{bmatrix} 0 & 1 & 8 & 6 & 0 & \frac{5}{2} & 0 \\ 0 & 0 & \frac{11}{2} & 5 & 3 & 2 & 2 \\ 1 & 0 & 0 & 2 & 8 & -\frac{1}{2} & 7 \\ \frac{7}{2} & \frac{11}{2} & 0 & 0 & 4 & -4 & \frac{17}{2} \\ \frac{9}{2} & \frac{17}{2} & 2 & 0 & 0 & -5 & \frac{15}{2} \\ \frac{3}{2} & \frac{7}{2} & 6 & 4 & 0 & 0 & \frac{5}{2} \\ 2 & 6 & \frac{19}{2} & 5 & -5 & 0 & 0 \end{bmatrix}$$

$$csm := q^7 - \frac{693}{4} q^5 - \frac{4655}{2} q^4$$

$$q^6 + \frac{35}{2} q^5 + 133 q^4$$

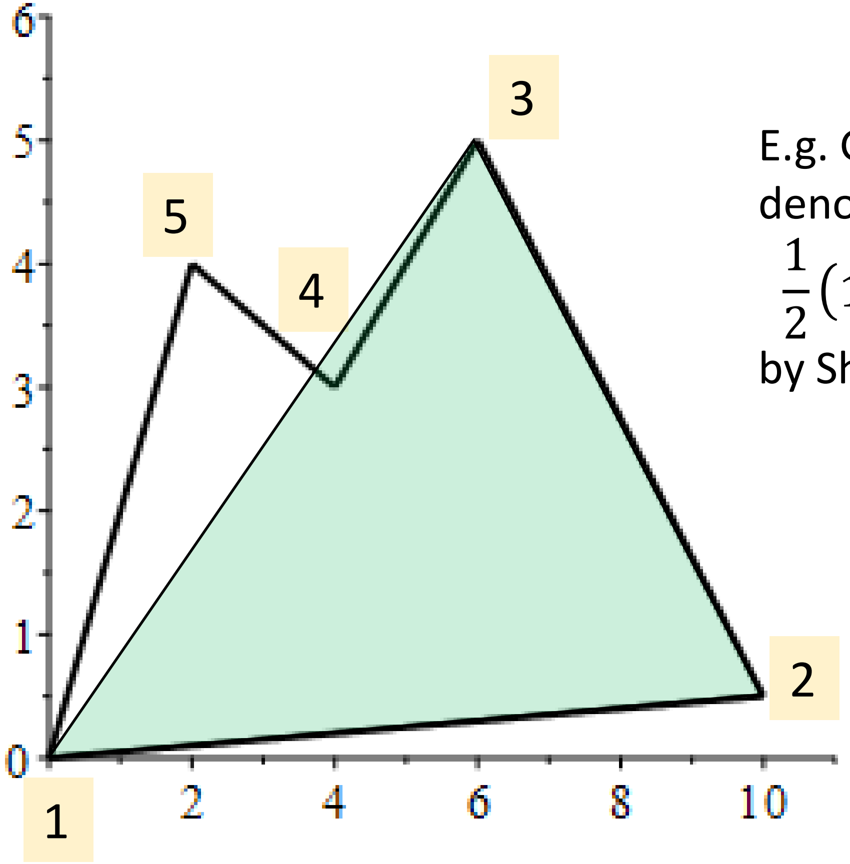
$$0, 0, 0, 0, \frac{35}{2}, -\frac{35}{4} - \frac{I\sqrt{903}}{4}, -\frac{35}{4} + \frac{I\sqrt{903}}{4}$$

The characteristic polynomial puzzle

Characteristic polynomial of triangle areas matrix for n-vertex polygon with area P is (apparently)

$$\det(A - qI) = q^{n-3}(q - P)(q + P/2 \pm \alpha i)$$

What is α?



E.g. Green triangle area denoted Δ_{12} is

$$\frac{1}{2}\left(10 \times 5 - 6 \times \tfrac{1}{2}\right) = {}^{47}/_{2}$$

by Shoelace formula.

Triangle areas matrix $(\Delta_{ij})_{\substack{i=1\dots n \\ j=i+1\dots n-2+i}}$

$$\begin{bmatrix} 0 & \frac{47}{2} & -1 & 5 & 0 \\ 0 & 0 & \frac{17}{2} & -\frac{1}{2} & \frac{39}{2} \\ \frac{47}{2} & 0 & 0 & -3 & 7 \\ 14 & \frac{17}{2} & 0 & 0 & 5 \\ \frac{39}{2} & 11 & -3 & 0 & 0 \end{bmatrix}$$

Char poly: $q\left(q - \frac{55}{2}\right)\left(q + \frac{55}{4} \pm \frac{\sqrt{5303}}{4} i\right)$