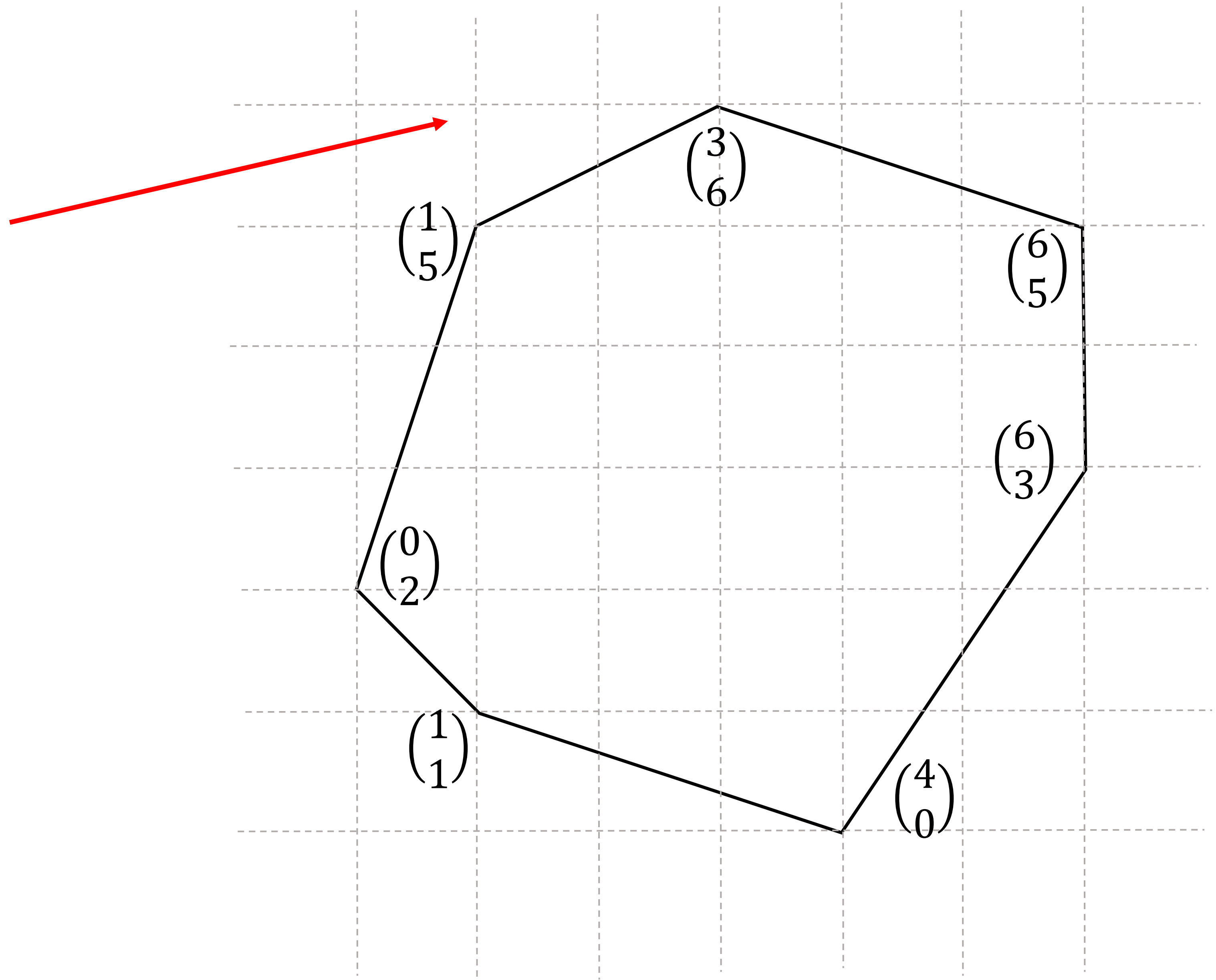
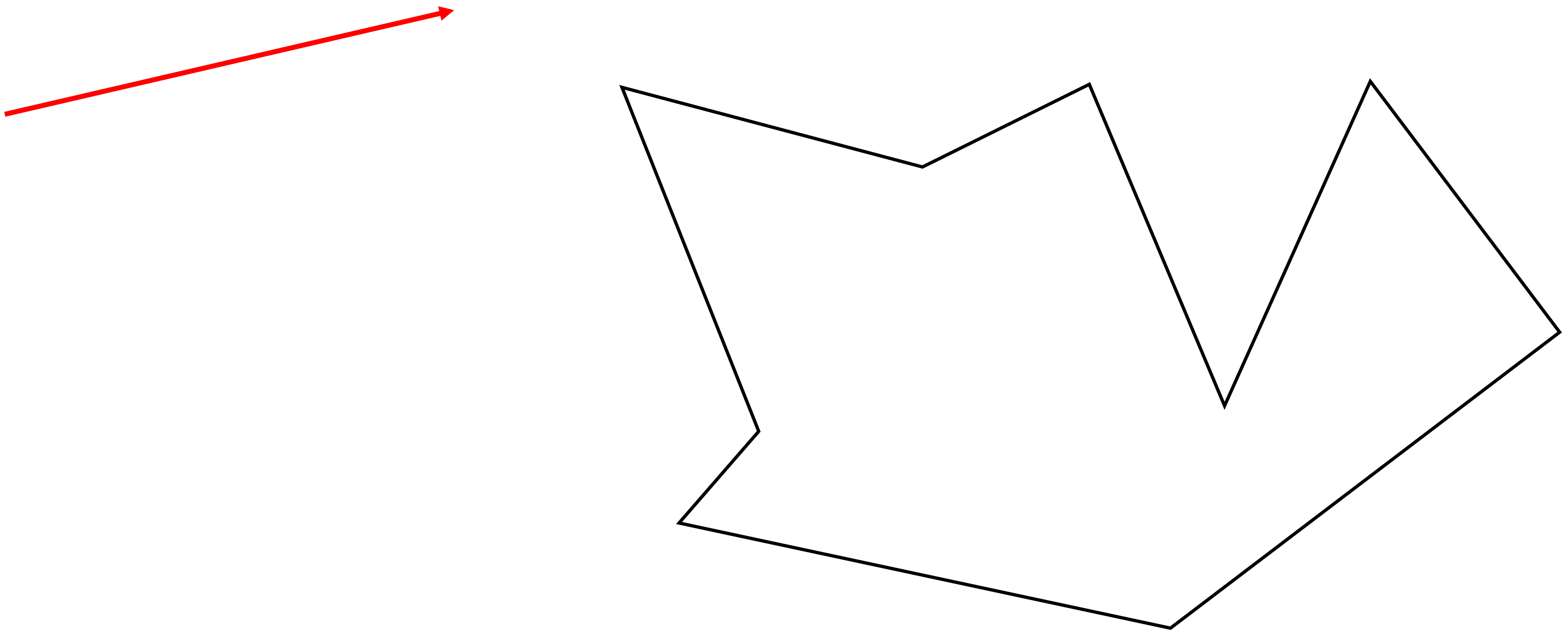
Bisecting convex polygons



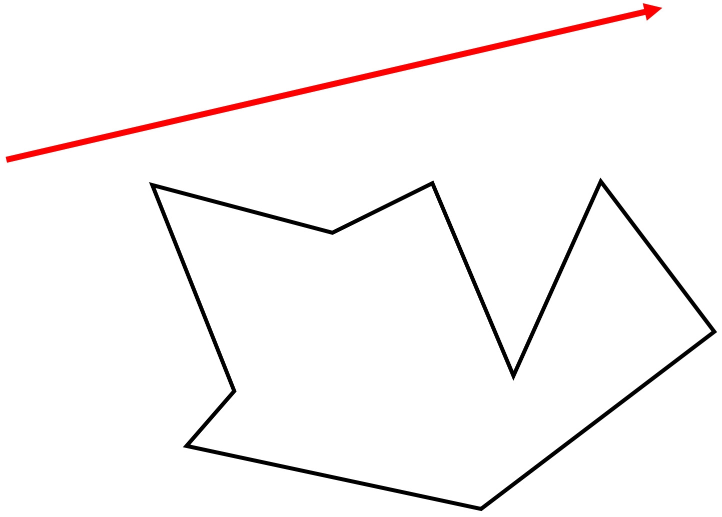
The problem I would really like to solve:

Given any simple connected polygon *P* (specified in terms of its coordinates) and a direction vector ***u***, write down the equation of the straight line bisecting the area of *P* in the direction of ***u***.

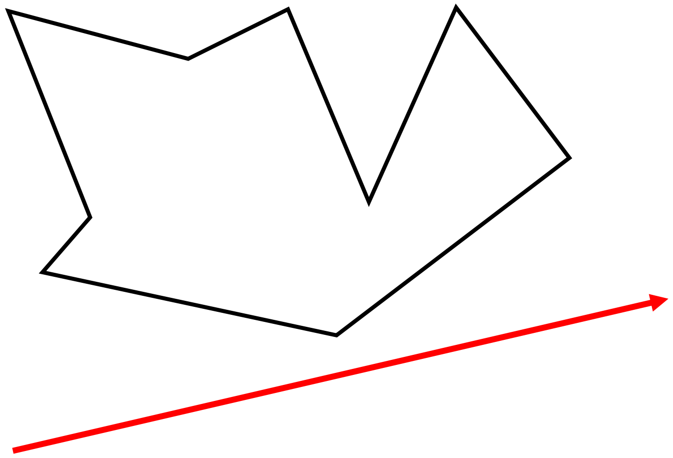


Which we know can be done…

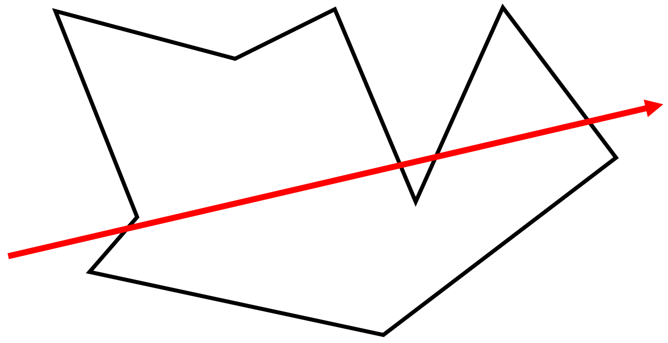
… by the intermediate value theorem…



Area 'above' line = 0



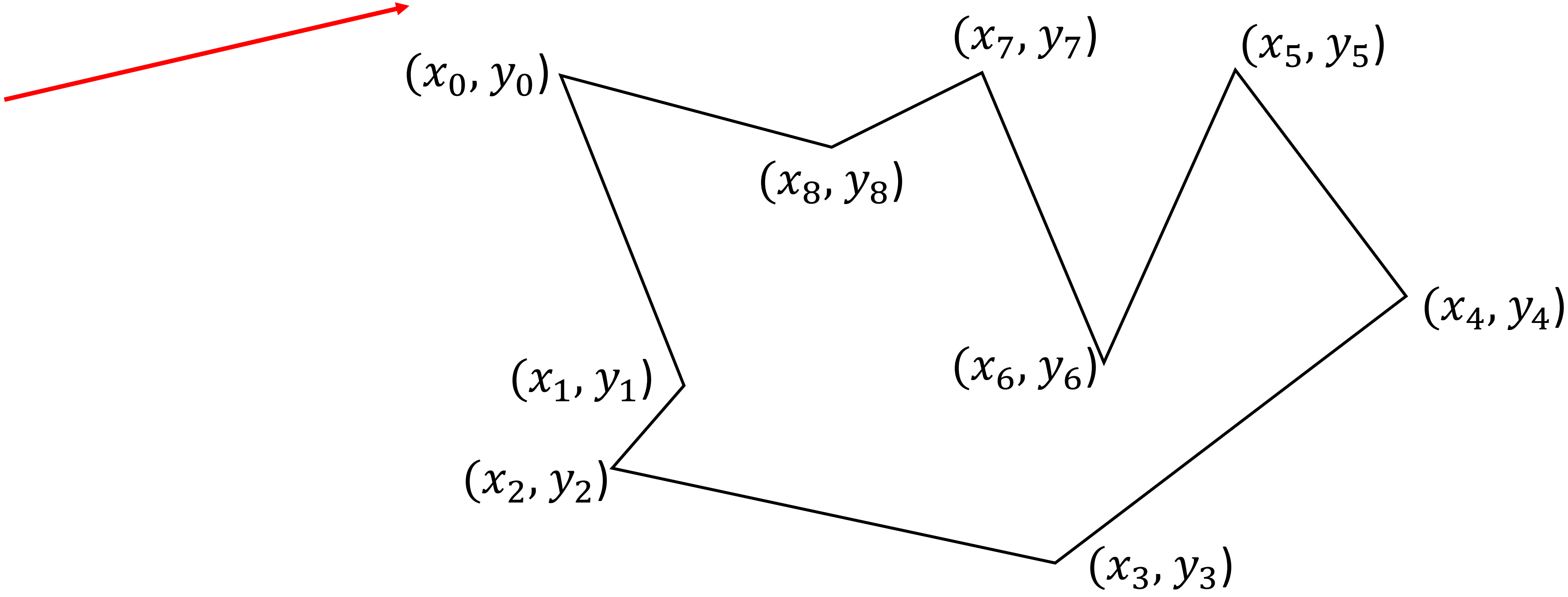
Area 'above' line = A (total area)



Area 'above' line = $A/2$

A restricted problem

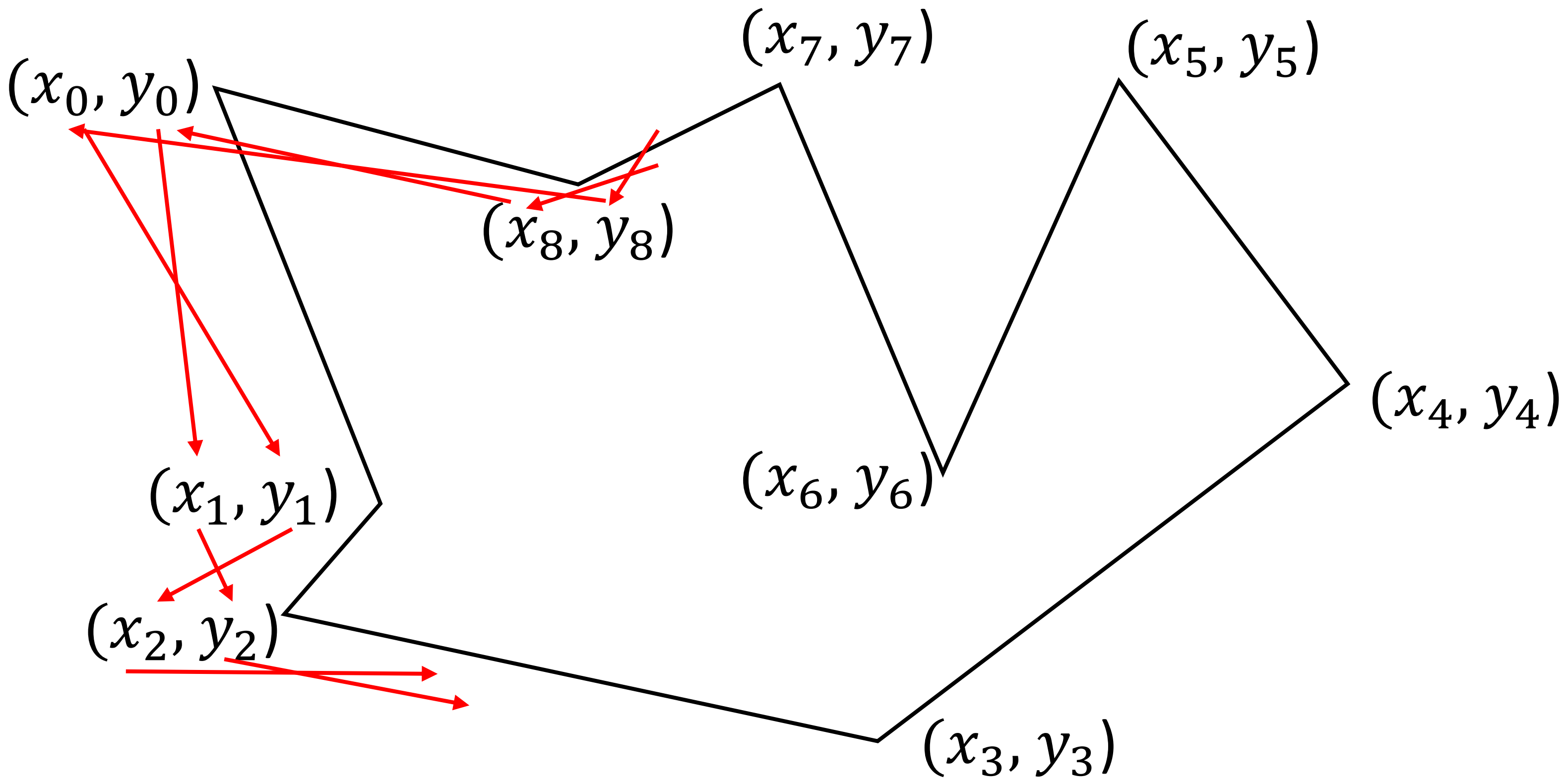
Given any simple connected polygon *P*, specified in terms of **rational** coordinates (x_i, y_i), and a direction vector ***u***, specified as a **rational number** (its slope), write down the equation of the straight line bisecting the area of *P* in the direction of ***u.***



The shoelace formula

Given any simple connected polygon P, specified in terms of coordinates (x_i, y_i), the area A of P is given by:

$$\frac{1}{2}(x_0y_1 - x_1y_0 + \ \ldots + x_{n-1}y_0 - x_0y_{n-1}).$$

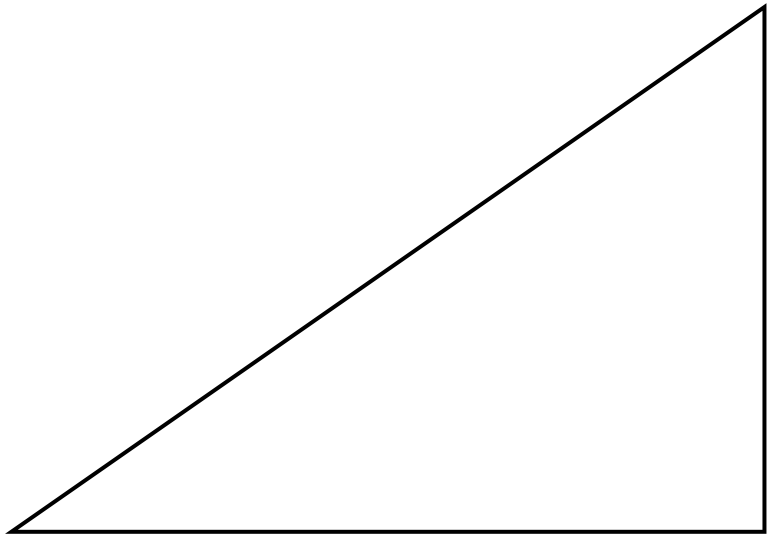


IVT and constructiveness

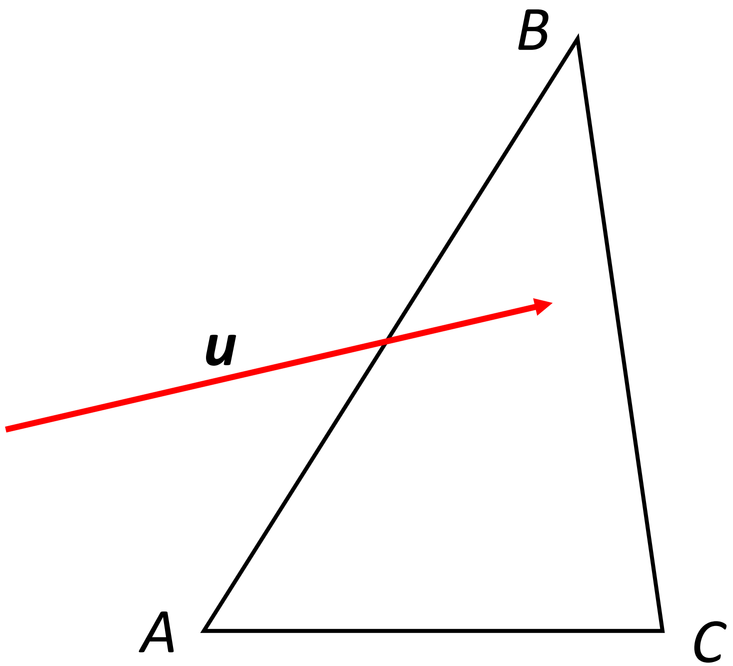
The shoelace formula says our restricted problem is solved by a straight line equation with rational coefficients.

But IVT will not necessarily produce this equation, only a rational solution arbitrarily close to it.

Area problems in the rationals are not necessarily straightforward! E.g. (Don Zagier) the 'simplest' right triangle with area 157 is:



Triangles: the easy case



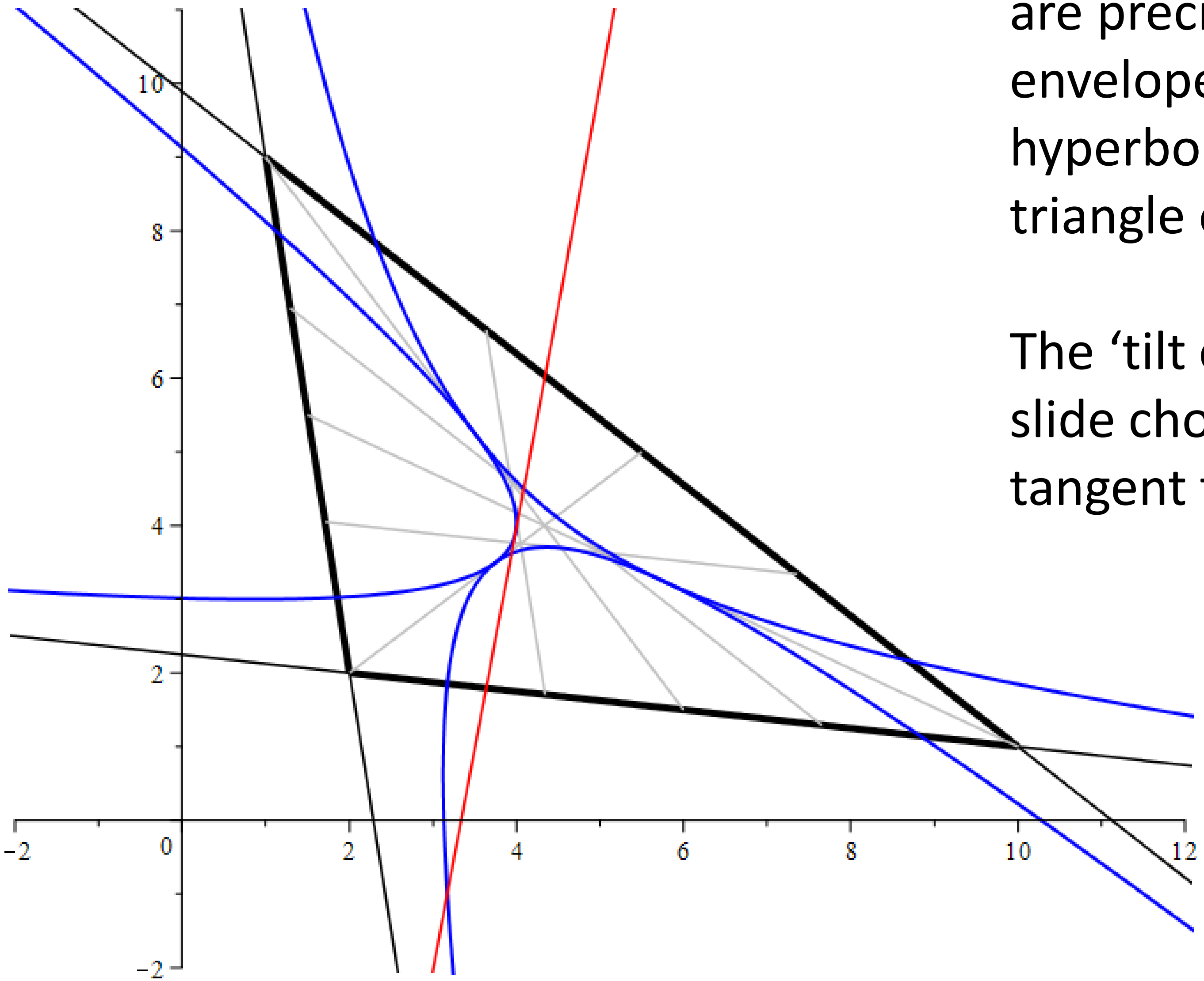
Bisecting straight line:

$$\boldsymbol{r} = \boldsymbol{u}\lambda + \begin{cases} (1-m)A + (1-t)mC + tmB & \text{for tilt on } BC \\ (1-m)B + (1-t)mA + tmC & \text{for tilt on } CA\,, \\ (1-m)C + (1-t)mB + tmA & \text{for tilt on } AB \end{cases}$$

where $t = \left(1 + \sqrt{\frac{2-w}{1+w}}\right)^{-1}, m = 1/\sqrt{2}\,,$

and w is unique solution in [0,1] to tilt equations.

Bisection envelopes (triangles)



For a triangle the bisecting lines are precisely the tangents to an envelope formed from three hyperbolae (which approach the triangle edges asymptotically).

The ‘tilt equations’ on the previous slide choose which hyperbola to be tangent to for a given direction ***u***.

Bisection envelopes (polygons)

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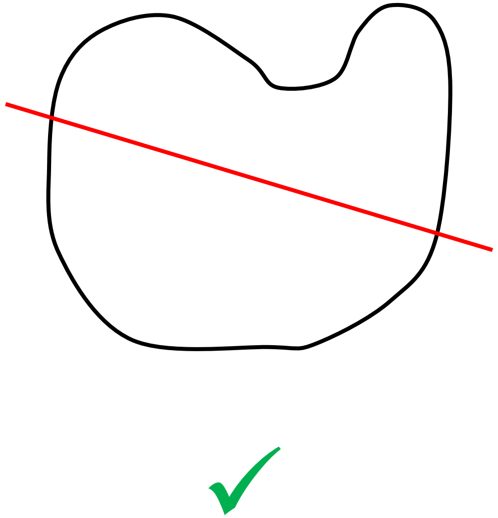
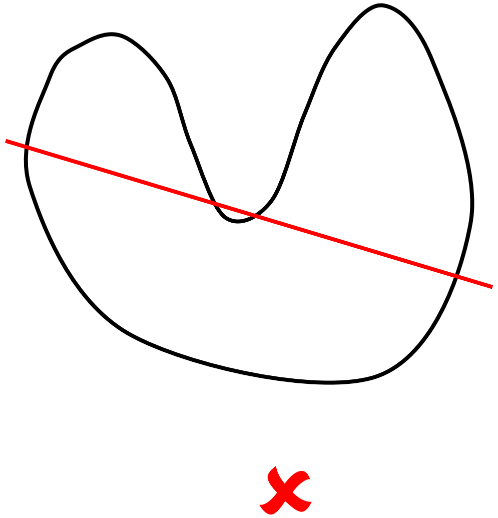
Bisection-convex: any bisecting straight line intersects the curve in exactly two points





Bisection envelopes

Noah Fechtor-Pradines

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Proposition 3.3. *The bisection envelope $\mathcal{B}$ of a polygon $\mathcal{S}$ is the union of a finite number of sections of hyperbolas. Let the set of all asymptotes of these hyperbolas be H, and let the set of all lines that contain the sides of $\mathcal{S}$ be G. Then $H \subseteq G$, with equality if no two lines in G are parallel.*

Strictly bisection-convex curves

We now restrict the class of curves $\mathcal{S}$ to be studied.

Definition 2.2. Define $\mathcal{S}$ and $\mathcal{L}$ as above. We say that $\mathcal{S}$ is *bisection convex* if for all θ, l_θ intersects $\mathcal{S}$ in exactly two points. Alternatively, for every point A on $\mathcal{S}$, there exists a unique point B also on $\mathcal{S}$ such that the line AB bisects the interior area of $\mathcal{S}$.

We also create a tighter restriction.

Definition 2.3. Define $\mathcal{S}$ and $\mathcal{L}$ as before. We say that $\mathcal{S}$ is *strictly bisection convex* if it is bisection convex and for all θ, l_θ is not tangent to $\mathcal{S}$. At any point where there are two tangents to $\mathcal{S}$ — one from each side — the l_θ through that point is distinct from both tangents.

Henceforth, unless otherwise stated, *it is assumed that $\mathcal{S}$ is strictly bisection convex.*

Bisection envelopes
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That IVT 2-pancakes issue again…

Define $A(\theta)$ and $B(\theta)$ to be the endpoints of the bisecting chord in direction θ, with $B(\theta) = A(\theta + \pi)$. We distinguish between $A(\theta)$ and $B(\theta)$ by demanding that for each point $Q \neq A(\theta), B(\theta)$ on the bisecting chord, the vector $A(\theta) - Q$ points in positive direction θ and the vector $B(\theta) - Q$ points in positive direction $\theta + \pi$.

Proposition 2.4. *Assume that S is bisection convex. Then $A(\theta)$ varies continuously with θ.*

Proof. First, we note that any two bisecting chords must intersect in the interior of S, for if they did not, the interior of S would be split into three regions, one of which would have zero area, which does not make sense.

From this, we have $\lim_{\epsilon \to 0} l_{\theta+\epsilon} = l_\theta$, as the limit of the intersection point $l_{\theta+\epsilon} \cap l_\theta$ is bounded. This also implies that the limit as $\epsilon \to 0$ of the distance from $A(\theta + \epsilon)$ to the intersection point $l_{\theta+\epsilon} \cap l_\theta$ is bounded. Therefore, the limit as $\epsilon \to 0$ of the perpendicular distance from $A(\theta + \epsilon)$ to l_θ is zero.

We have that $\lim_{\epsilon \to 0} A(\theta + \epsilon)$ must be a point P on l_θ which intersects S, where for every other point Q on the bisecting chord with direction θ, the vector $P - Q$ points in positive direction θ. There is only one such point, $A(\theta)$; therefore,

$$\lim_{\epsilon \to 0} A(\theta + \epsilon) = A(\theta),$$

and $A(\theta)$ varies continuously with θ. □

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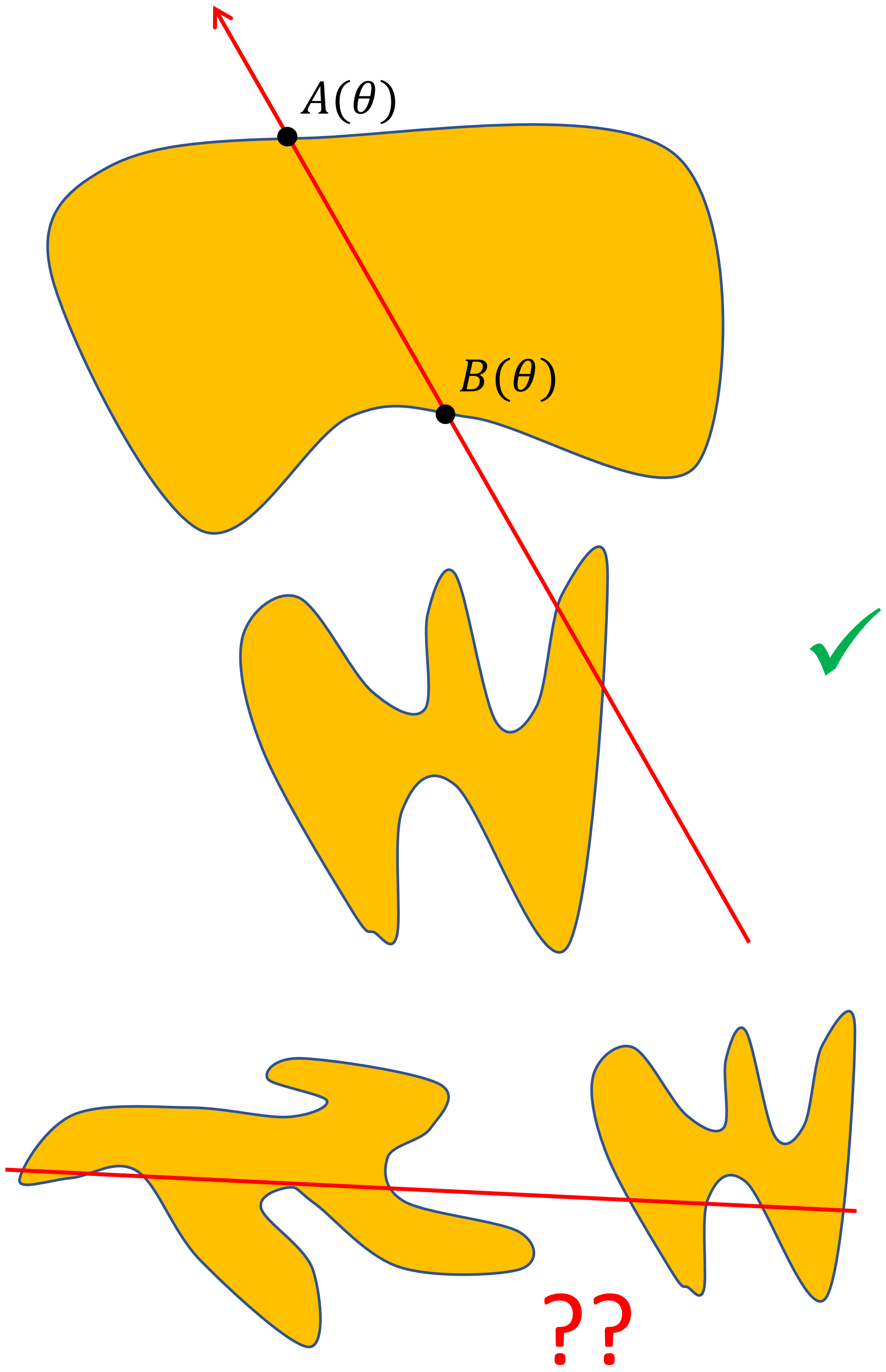
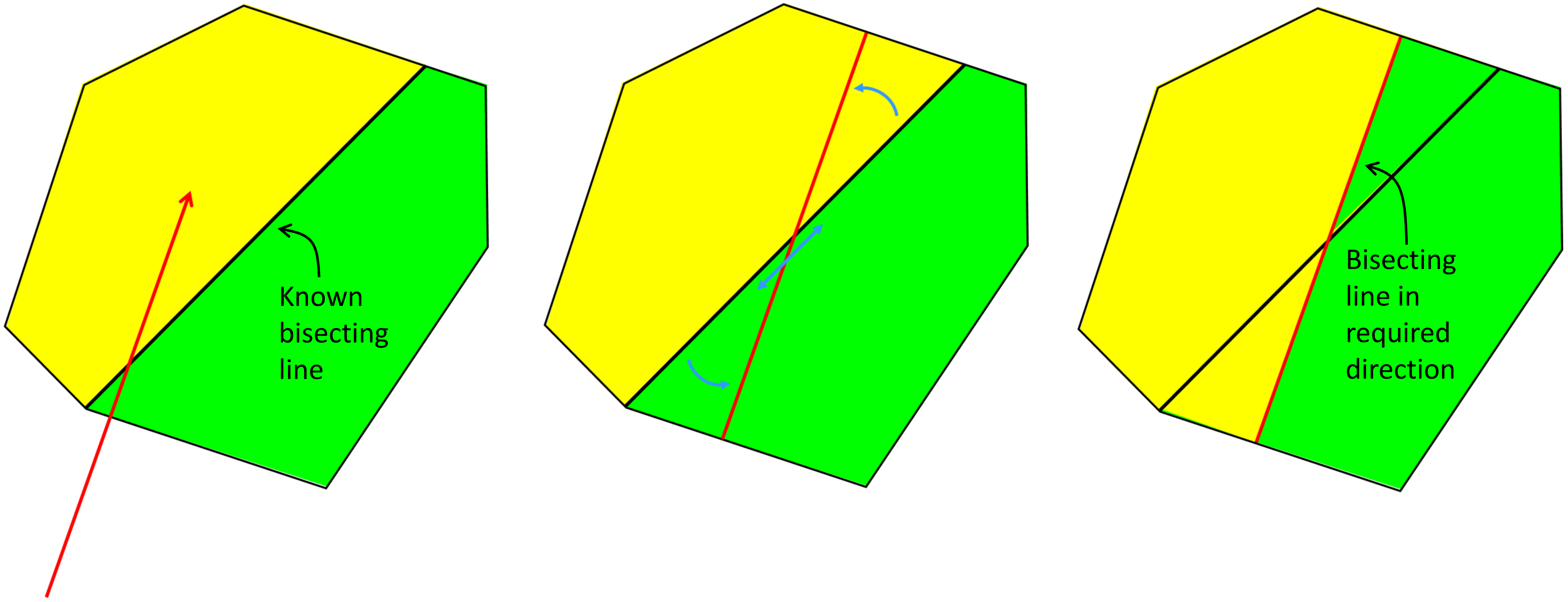


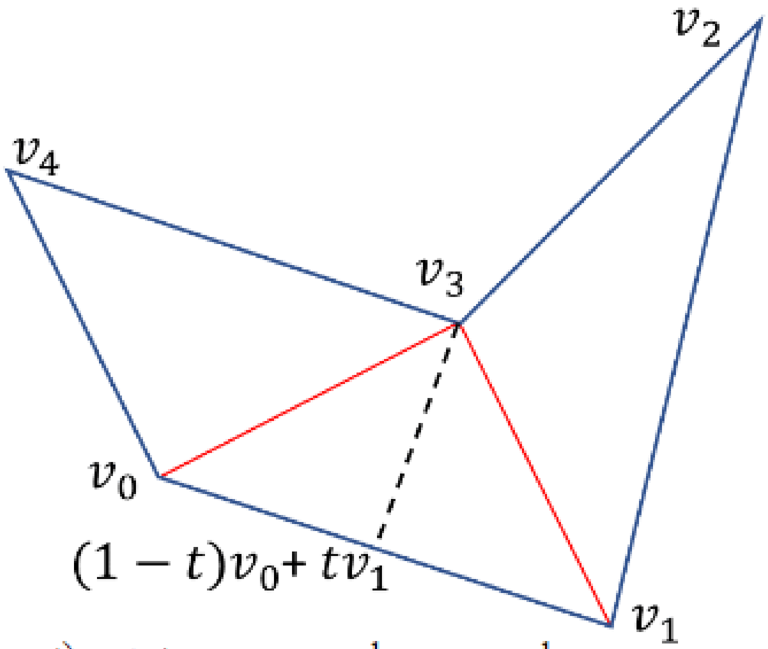
A straight line equation for convex polygon bisection…

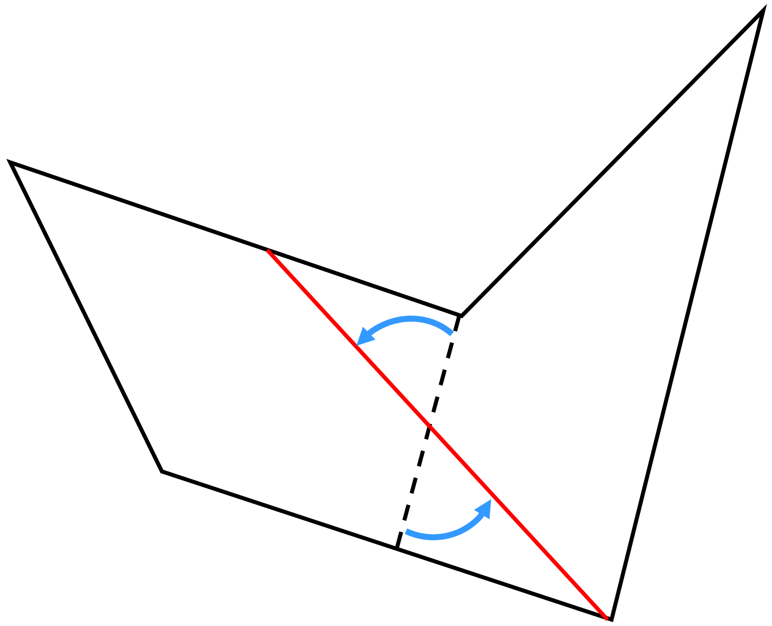


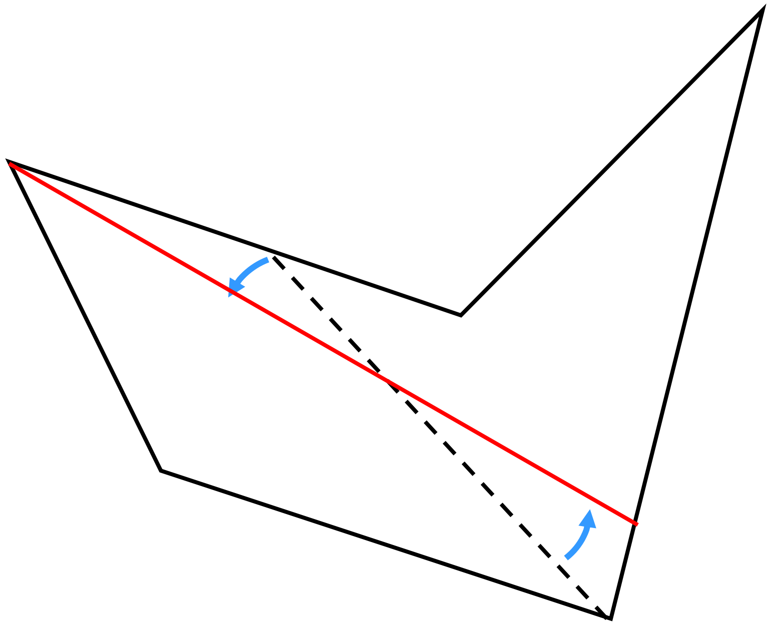
… or even bisection-convex polygon bisection?

An Application

We may triangulate a polygon on n vertices by adding $n-3$ diagonals, as illustrated on the right. We would like to test if some straight line joining a triangle vertex to the opposite polygon edge bisects the area of the polygon. In our diagram this requires a value of $t \in [0,1]$ for which the polygons $v_0, (1-t)v_0 + tv_1, v_3, v_4$ and $(1-t)v_0 + tv_1, v_1, v_2, v_3$ have equal area.







An application of the shoelace formula gives

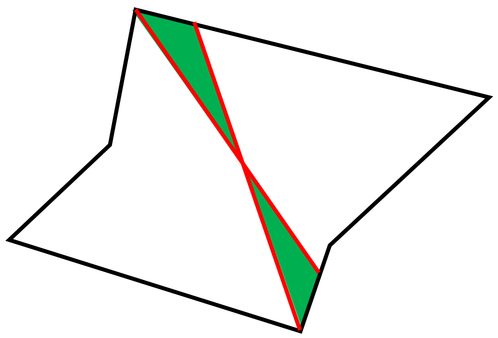
$$t = \frac{A_R - A_L}{2A_\Delta}$$

A_L = area to left of middle triangle

A_R = remaining polygon area

A_Δ = area of middle triangle

Strategy for writing down a bisecting straight line equation



Divide interior of polygon into sectors bordered by pairs of bisecting lines ***x*** and ***y***

Set of chordal lines in sector of ***x*** and ***y*** is specified as

$$\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) = w\boldsymbol{x} + (1-w)\boldsymbol{y}, w \in [0,1]$$

If given direction vector ***u*** lies in sector then this will solve to give $w \in [0,1]$

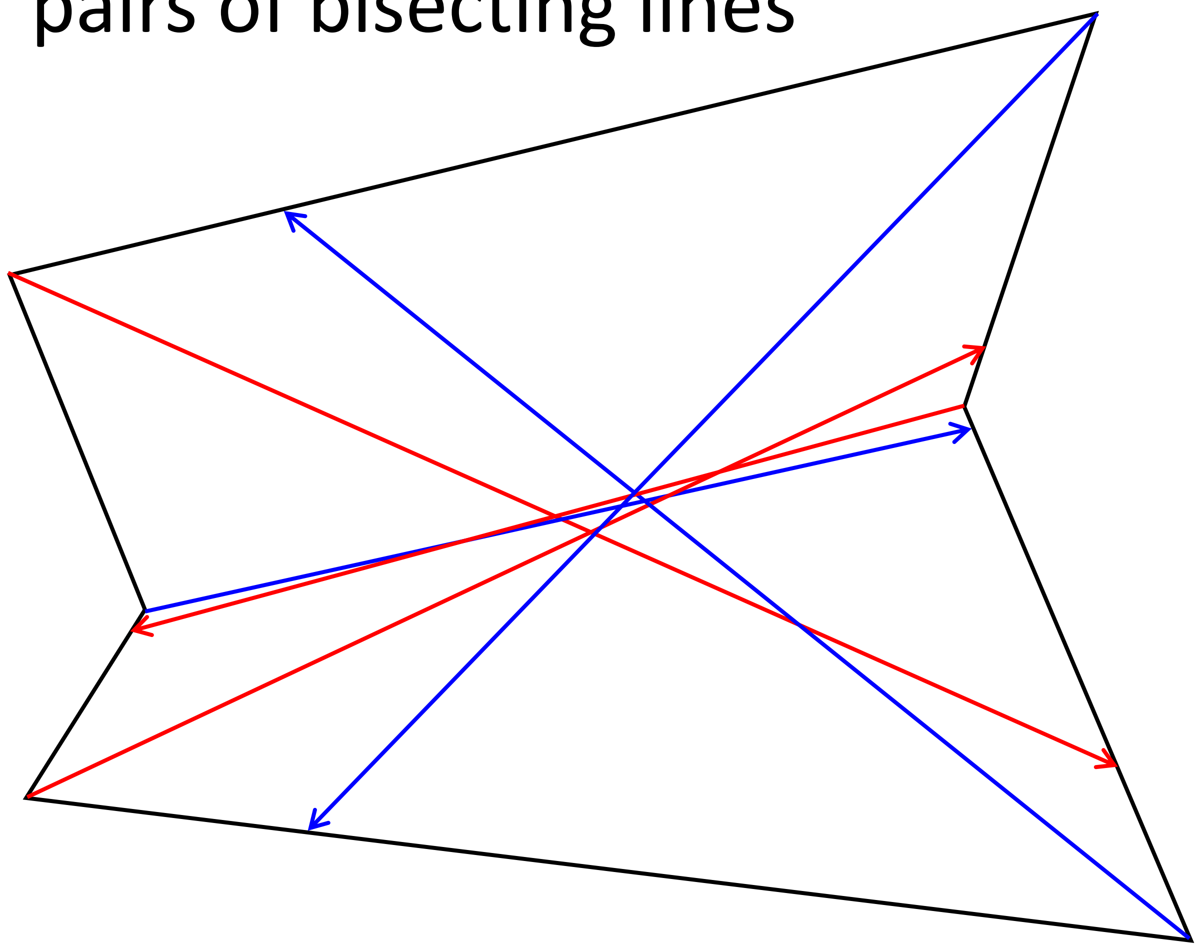
Point on ***x*** where bisecting line will pass is $t\boldsymbol{x}$, where $t = \left(1 + \sqrt{T(w)}\right)^{-1}, w \in [0,1]$

For triangle we had $t = \left(1 + \sqrt{\frac{2-w}{1+w}}\right)^{-1}$;

For trapezoid we had $t = \left(1 + \sqrt{\frac{2-(1-\gamma^2)w}{1+\gamma^2+(1-\gamma^2)w}}\right)^{-1}$, for tilting parallel to base, top base = bottom base scaled by γ

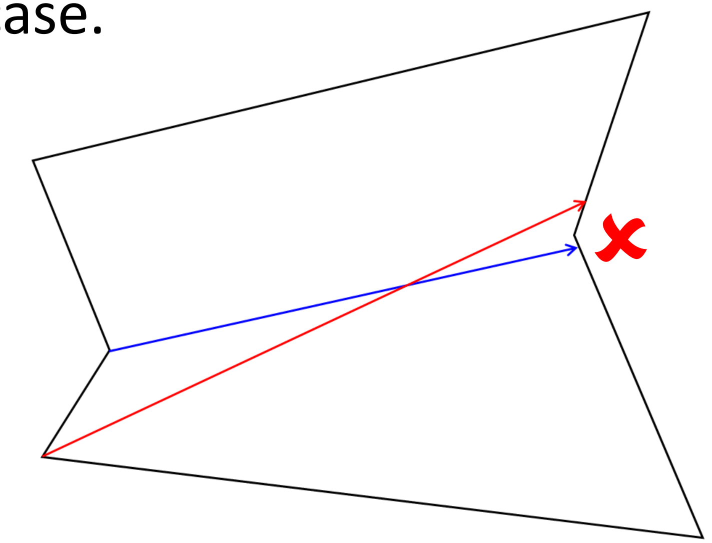
In general case things may not simplify so nicely (i.e. *T(w)* will involve ***u***, ***x*** and ***y***).

First step: Divide interior of polygon into sectors bordered by pairs of bisecting lines



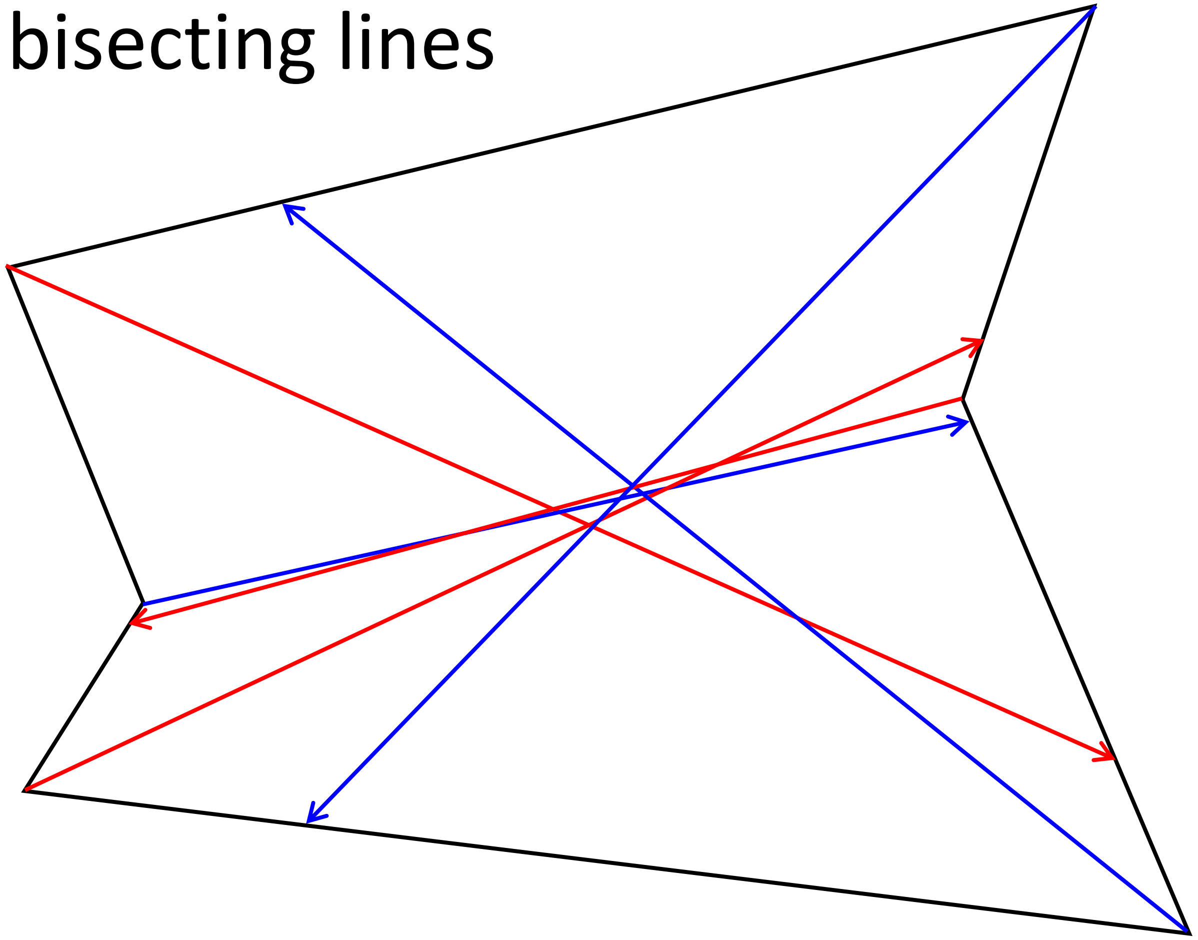
Apply the bootlace formula trick at each vertex.

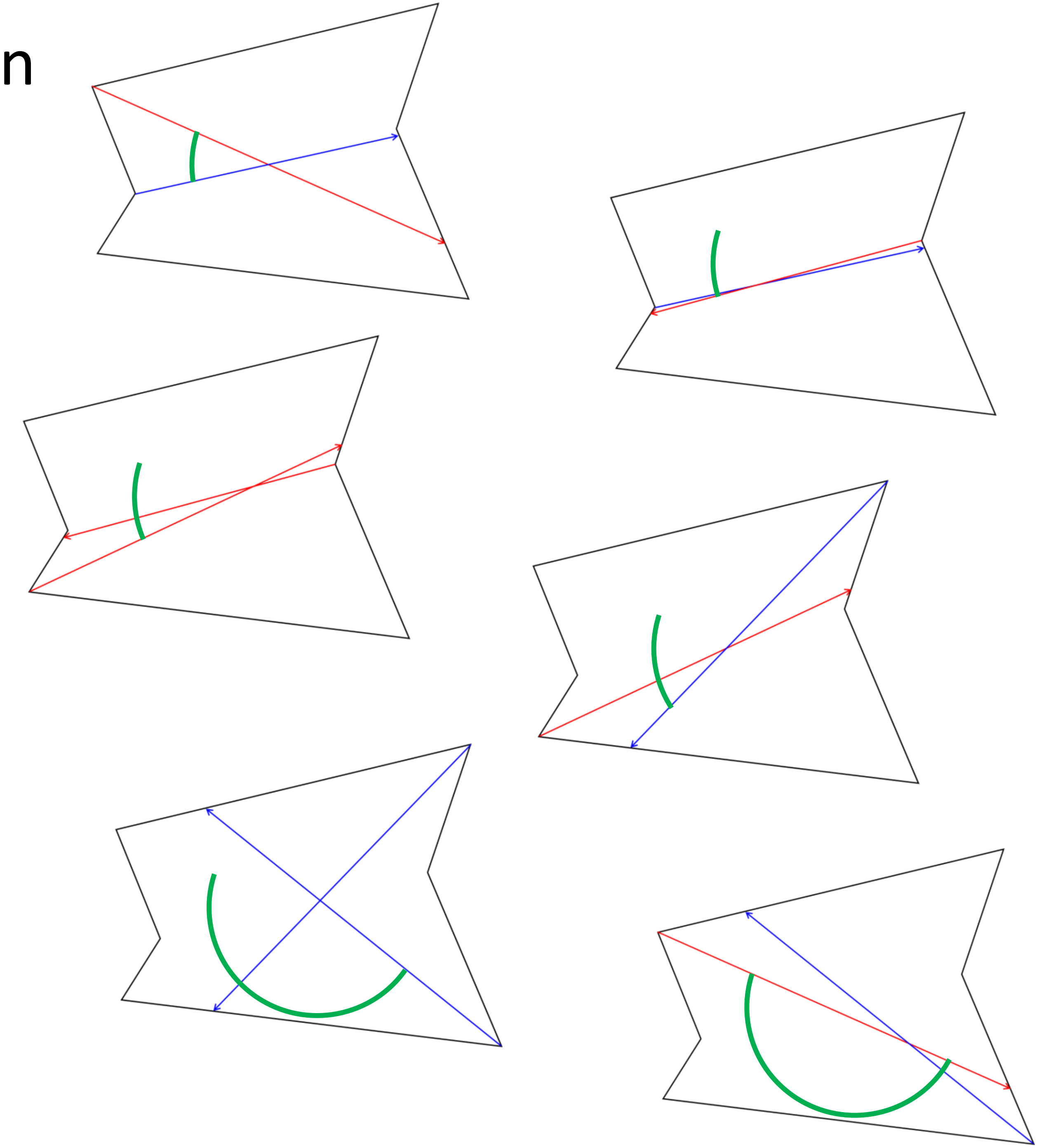
Pairs of lines defining a sector cannot ‘include’ a vertex because tilting won’t work in this case.



In fact even $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) = w\boldsymbol{x} + (1 - w)\boldsymbol{y}$, won’t work.

First step: Divide interior of polygon into sectors bordered by pairs of bisecting lines





Q. Will this always cover an angle of $\tau/2$ for bisection-convex polygons?