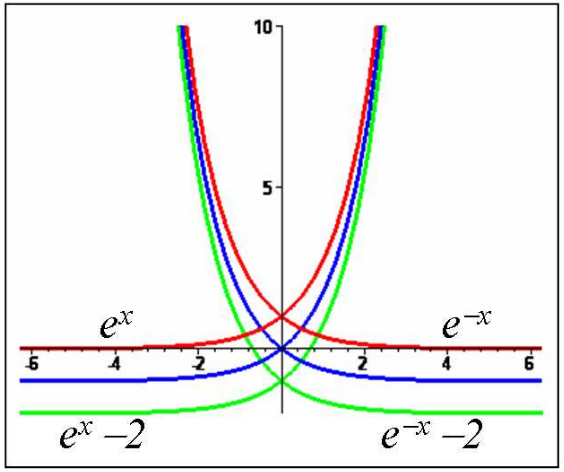
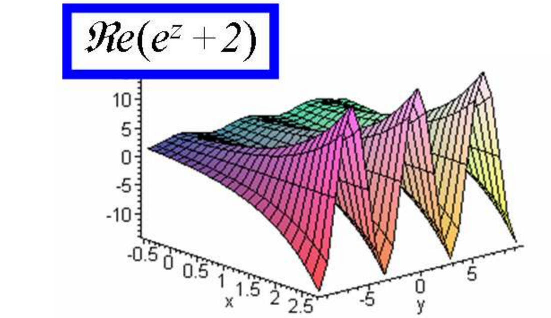
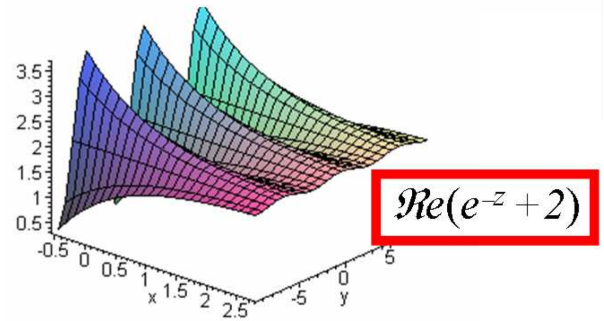
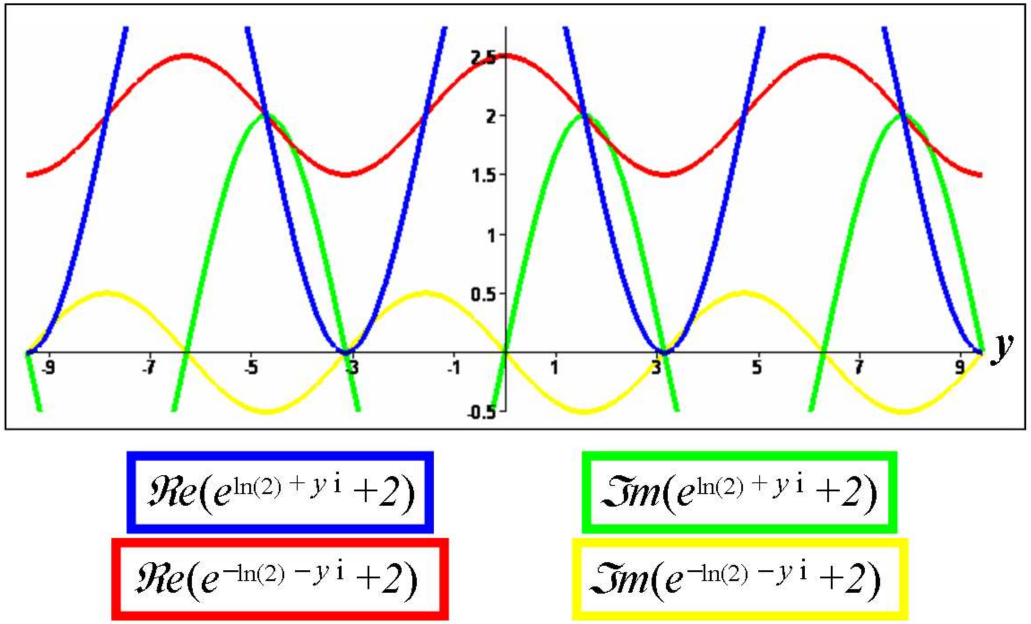
THEOREM OF THE DAY

Nevanlinna's Five-Value Theorem *If f and g are meromorphic functions of one complex variable and f and g have the same inverse images (ignoring multiplicities) on five distinct values then they are identically equal.*









Mostly, we may think of meromorphic functions as sums of rational functions (ratios of polynomials) and entire functions (well-behaved: polynomials, e^x, $\sin(x), \ldots$). Two such functions, $f(x)$ and $g(x)$, have the same inverse image on a value a, or are said to *share* the value a, if $f(x) - a = 0$ and $g(x) - a = 0$ have the same solutions (*zeros*). Now, above left, the upper, red, curves are e^x and e^{-x}, clearly different. But surely $e^x - a$ and $e^{-x} - a$ have the same zeros for *infinitely* many values of a?! Well, they don't share the value +2, since the bottom, green curves cut the x axis at different points. But they share +1 (the middle, blue, curve) and zero and, for all $a < 0$, we will get the same, empty, set of zeros. However, functions of a *complex* variable $z = x + iy$ are another matter! The real parts of $e^z - a$ and $e^{-z} - a$, for $a = -2$, are shown in the graphs above centre, their periodic nature being explained by the identity $e^{x+iy} = e^x(\cos y + i \sin y)$. On the right, a cross section has been taken at $x = \ln(2)$; only $e^{\ln(2)+iy} + 2$ intersects the complex plane in its real part (the blue line; in fact, it *touches* the plane, giving a zero of multiplicity two, but the theorem ignores multiplicities). The imaginary parts of both curves intersect the plane at these same points, but for a shared value we would need green meets blue *and* red meets yellow. We conclude that $e^z + 2$ has zeros of the form $\ln(2) + iy$ which are *not* zeros of $e^{-z} + 2$: therefore -2 is *not* shared. Still, e^z and e^{-z} share four values: 0, 1, -1 and, in a technical sense, ∞. So the 'five' in the theorem cannot be reduced to a 'four'.

This 1926 result of the Finnish mathematician Rolf Nevanlinna (1895–1980) was described by the distinguished analyst Lee Rubel as his favourite in all mathematics. It follows from the second fundamental theorem of Nevanlinna Theory, in turn described by the mighty Hermann Weyl as one of the greatest achievements of twentieth century mathematics.

Web link: www.theoremoftheday.org/Analysis/Nevanlinna5/Frank-Hua-Vaillancourt.pdf (in French and English).

Further reading: *Entire and Meromorphic Functions* by Lee A. Rubel and James E. Colliander, Springer-Verlag New York, 1996.