**THEOREM OF THE DAY**

The Riemann Rearrangement Theorem If \(\sum_{k=0}^{\infty} a_k\) is a series which is conditionally convergent, and \(c\) is any real number, then the terms of the series may be rearranged to give convergence to \(c\), i.e. there is a permutation \(\pi\) of the nonnegative integers such that \(\sum a_{\pi(k)} = c\).

**A** Partial sums of rearrangements of Leibniz: \(\sum_{k=0}^{\infty} (-1)^k/(2k+1)\).

**B** Partial sums of positive and negative term series.

**C** Rearranging Leibniz to give Liouville.

**D** Zooming in at around \(\sum_{k=0}^{\infty} a_{\pi(k)}\), \(n \geq 400\).

A: Leibniz’ series converges to \(\pi/8 \approx 0.7854\) conditionally; but not absolutely because \(\sum |(-1)^k/(2k+1)|\) does not converge. Indeed, B: the positive terms give a divergent series, as do the negative terms. Interleaving subseries of these divergent series can give convergence to any value. We have chosen C: Liouville’s number (in 1851, the first ever shown to be transcendental). Every time a positive term is incorporated it is followed by the least number of negative terms needed to bring the partial sum back down below Liouville (twelve are required to reduce 1 to \(< 1.110...\)).

Riemann’s 1854 habilitation thesis assembled a whole workshop of new tools, among them this classic analysis of divergence, for investigating the behaviour of functions represented by the then-still-controversial trigonometric series of Joseph Fourier.

**Web link:** divien2.wordpress.com/2011/05/21/rearrangement-theorem/

**Further reading:** *A Radical Approach to Real Analysis, 2nd edition* by David M. Bressoud, Mathematical Association of America, 2007, chapter 5.