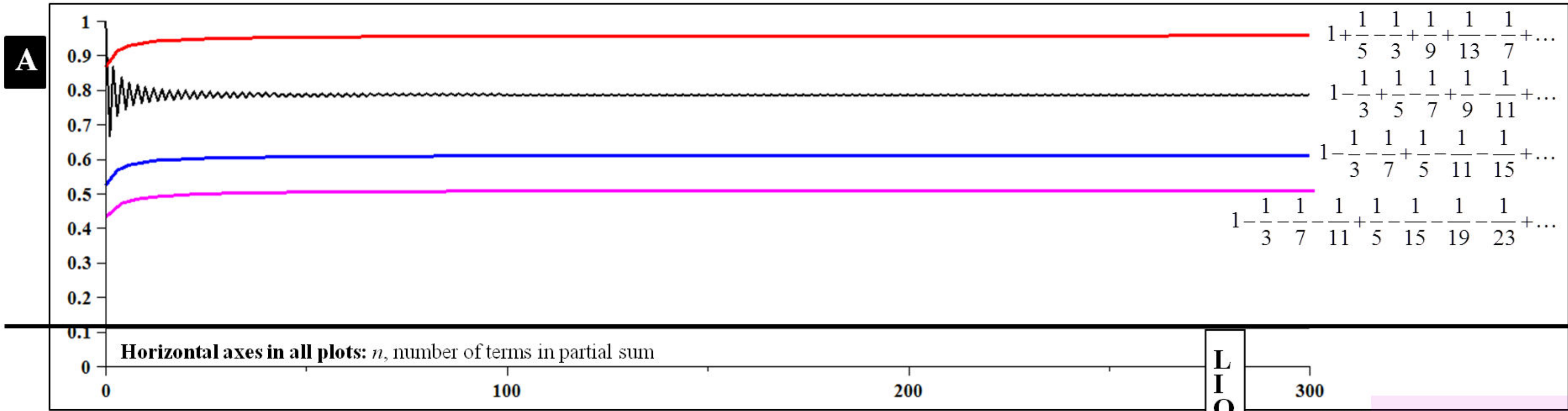
THEOREM OF THE DAY

The Riemann Rearrangement Theorem *If* $\sum_{k=0}^{\infty} a_k$ *is a series which is conditionally convergent, and c is any real number, then the terms of the series may be rearranged to give convergence to c, i.e. there is a permutation* π *of the nonnegative integers such that* $\sum a_{\pi(k)} = c$.

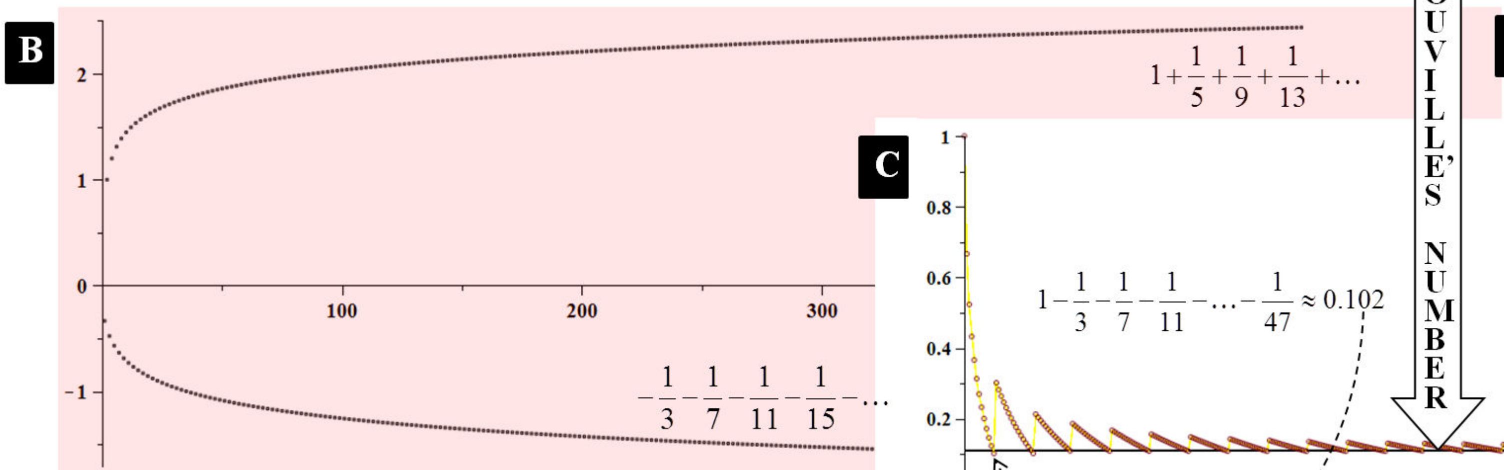


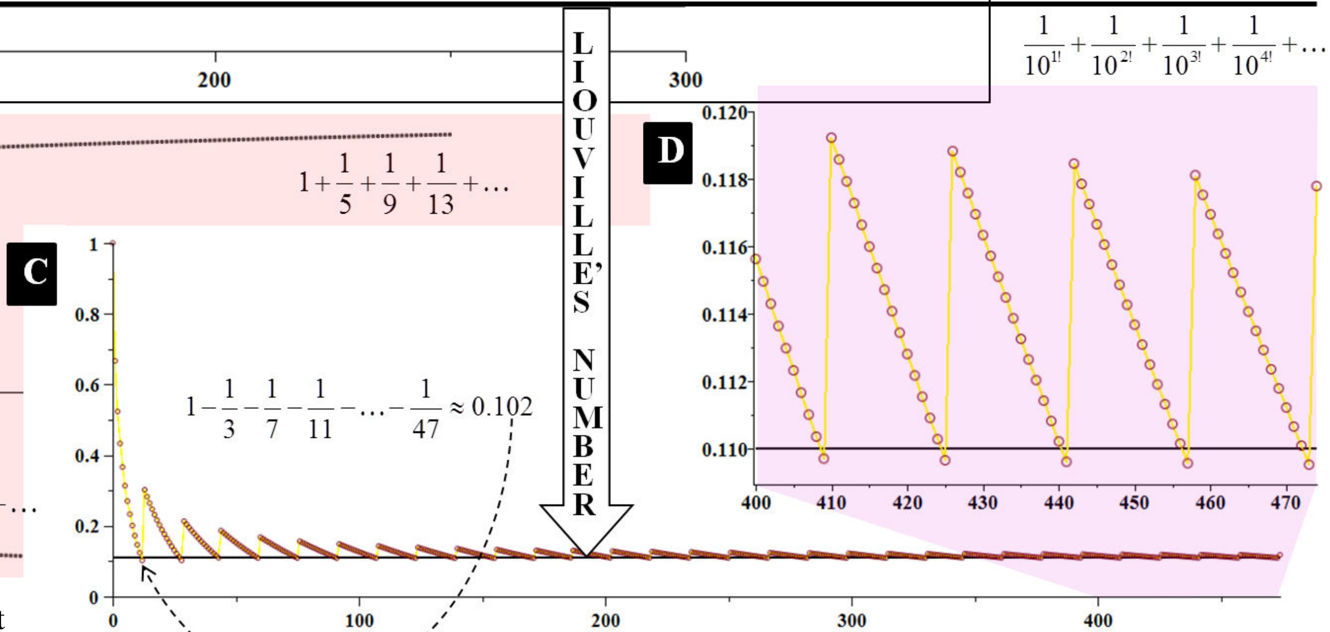
A Partial sums of rearrangements of Leibniz: $\sum_{k=0}^{\infty}(-1)^k/(2k+1)$.

B Partial sums of positive and negative term series.

C Rearranging Leibniz to give Liouville.

D Zooming in at around $\sum_{k=0}^{n} a_{\pi(k)}, n \geq 400$.





A: Leibniz' series converges to $\tau/8 \approx 0.7854$ conditionally; but not absolutely because $\sum \left|(-1)^k/(2k+1)\right|$ does not converge. Indeed, **B:** the positive terms give a divergent series, as do the negative terms. Interleaving subseries of these divergent series can give convergence to any value. We have chosen **C:** Liouville's number (in 1851, the first ever shown to be transcendental). Every time a positive term is incorporated it is followed by the least number of negative terms needed to bring the partial sum back down below Liouville (twelve are required to reduce 1 to < 1.110...).

Riemann's 1854 habilitation thesis assembled a whole workshop of new tools, among them this classic analysis of divergence, for investigating the behaviour of functions represented by the then-still-controversial trigonometric series of Joseph Fourier.

Web link: divien2.wordpress.com/2011/05/21/rearrangement-theorem/.

Further reading: *A Radical Approach to Real Analysis, 2nd edition* by David M. Bressoud, Mathematical Association of America, 2007, chapter 5.