



# THEOREM OF THE DAY

**The Remainder Theorem** If a polynomial  $f(x)$  is divided by  $(x - \alpha)$  then the remainder is  $f(\alpha)$ .

**Corollary (The Factor Theorem)** A polynomial  $f(x)$  has  $(x - \alpha)$  as a factor if and only if  $f(\alpha) = 0$ .

$$\begin{array}{r}
 \text{quotient} \\
 a_0x^2 + (\alpha a_0 + a_1)x + (\alpha(\alpha a_0 + a_1) + a_2) \\
 \hline
 x - \alpha \overline{) a_0x^3 + a_1x^2 + a_2x + a_3} \\
 \underline{a_0x^3 - \alpha a_0x^2} \quad \text{subtract!} \\
 (\alpha a_0 + a_1)x^2 + a_2x \\
 \underline{(\alpha a_0 + a_1)x^2 - \alpha(\alpha a_0 + a_1)x} \\
 (\alpha(\alpha a_0 + a_1) + a_2)x + a_3 \\
 \underline{(\alpha(\alpha a_0 + a_1) + a_2)x - \alpha(\alpha(\alpha a_0 + a_1) + a_2)} \\
 \alpha(\alpha(\alpha a_0 + a_1) + a_2) + a_3 \\
 \hline
 \text{remainder} \\
 a_0\alpha^3 + a_1\alpha^2 + a_2\alpha + a_3 = f(\alpha)
 \end{array}$$

The diagram illustrates the long division of a cubic polynomial  $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3$  by the binomial  $x - \alpha$ . The quotient is  $a_0x^2 + (\alpha a_0 + a_1)x + (\alpha(\alpha a_0 + a_1) + a_2)$ . The remainder is  $\alpha(\alpha(\alpha a_0 + a_1) + a_2) + a_3$ , which is equal to  $f(\alpha)$ . Handwritten notes include 'quotient', 'numerator', 'denominator', 'subtract!', 'bring down!', and 'remainder'.

The Remainder Theorem follows immediately from the definition of polynomial division: to divide  $f(x)$  by  $g(x)$  means precisely to write  $f(x) = g(x) \times \text{quotient} + \text{remainder}$ . If  $g(x)$  is the binomial  $x - a$  then choosing  $x = \alpha$  gives  $f(\alpha) = 0 \times \text{quotient} + \text{remainder}$ . The illustration above shows the value  $f(\alpha)$  emerging as the remainder in the case where  $f(x)$  is a cubic polynomial and 'long division' by  $x - \alpha$  is carried out. The precise form in which the remainder is derived,  $\alpha(\alpha(\alpha a_0 + a_1) + a_2) + a_3$ , indicates a method of calculating  $f(\alpha)$  without separately calculating each power of  $\alpha$ ; this is effectively the content of *Ruffini's Rule* and the *Horner Scheme*. In the case where  $a_1$  is nearly equal to  $-\alpha a_0$ ;  $a_2$  is nearly equal to  $-\alpha(\alpha a_0 + a_1)$ , etc, this can be highly effective; try, for example, evaluating  $x^6 - 103x^5 + 396x^4 + 3x^2 - 296x - 101$  at  $x = 99$ : the answer (see p. 14 of [www.theoremoftheday.org/Docs/Polynomials.pdf](http://www.theoremoftheday.org/Docs/Polynomials.pdf)) comes out without having to calculate anything like  $99^6$  (a 12-digit number).

The Remainder and Factor theorems were surely known to Paolo Ruffini (1765–1822) who, modulo a few gaps, proved the impossibility of solving the quintic by radicals, and to William Horner (1786–1837); and probably well before that, to Descartes, who indeed states the Factor theorem explicitly in his *La Géométrie* of 1637. Polish school students learn about the Factor Theorem under the name "twierdzenie Bézout" (Bézout Theorem) after Etienne Bézout (1730–1783) but this attribution is obscure.

**Web link:** [eprints.soton.ac.uk/168861/](http://eprints.soton.ac.uk/168861/). The Polish nomenclature is discussed here: [pl.wikipedia.org/wiki/Twierdzenie\\_Bézouta](http://pl.wikipedia.org/wiki/Twierdzenie_Bézouta) (in Polish).

**Further reading:** *The Geometry of René Descartes*, 1925 annotated translation by David E. Smith and Marcia Latham, reprinted by Cosimo Classics, 2007 (in which copy the above citation is on p. 179).

