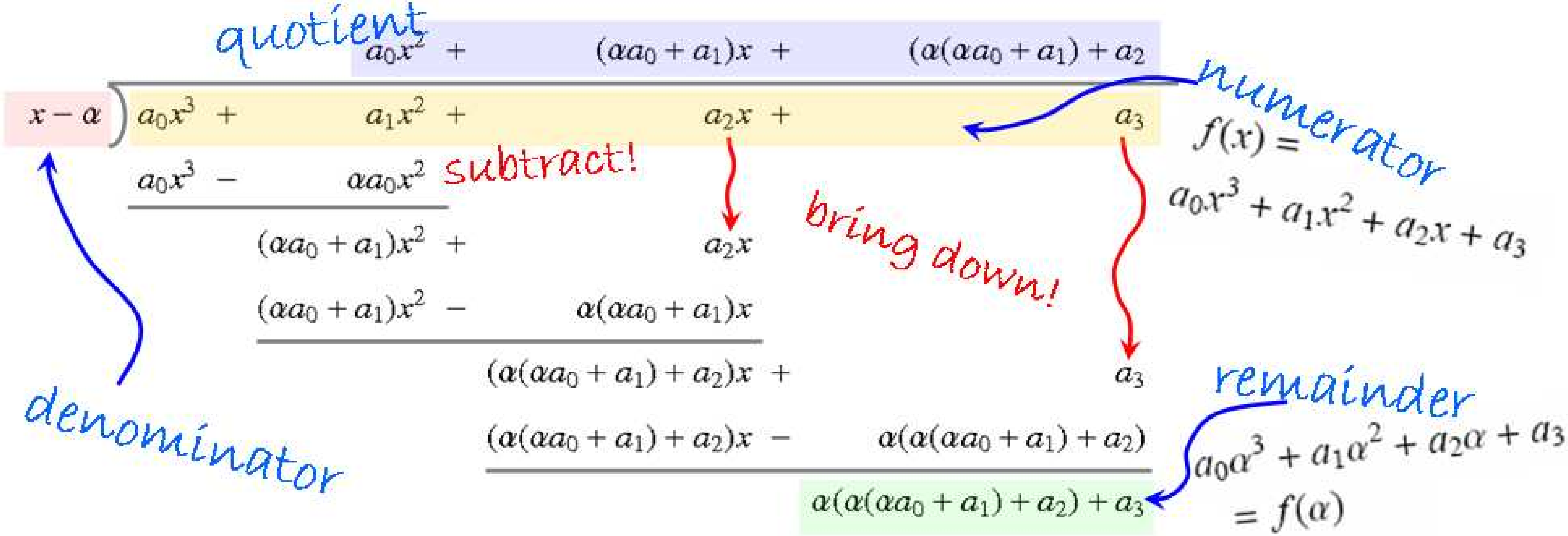
THEOREM OF THE DAY

The Remainder Theorem *If a polynomial* $f(x)$ *is divided by* $(x - \alpha)$ *then the remainder is* $f(\alpha)$.

Corollary (The Factor Theorem) *A polynomial* $f(x)$ *has* $(x - \alpha)$ *as a factor if and only if* $f(\alpha) = 0$.



The Remainder Theorem follows immediately from the definition of polynomial division: to divide $f(x)$ by $g(x)$ means precisely to write $f(x) = g(x) \times \text{quotient} + \text{remainder}$. If $g(x)$ is the binomial $x - a$ then choosing $x = \alpha$ gives $f(a) = 0 \times \text{quotient} + \text{remainder}$. The illustration above shows the value $f(\alpha)$ emerging as the remainder in the case where $f(x)$ is a cubic polynomial and 'long division' by $x - \alpha$ is carried out. The precise form in which the remainder is derived, $\alpha(\alpha(\alpha a_0 + a_1) + a_2) + a_3$, indicates a method of calculating $f(\alpha)$ without separately calculating each power of α; this is effectively the content of *Ruffini's Rule* and the *Horner Scheme*. In the case where a_1 is nearly equal to $-\alpha a_0$; a_2 is nearly equal to $-\alpha(\alpha a_0 + a_1)$, etc, this can be highly effective; try, for example, evaluating $x^6 - 103x^5 + 396x^4 + 3x^2 - 296x - 101$ at $x = 99$: the answer (see p. 14 of www.theoremoftheday.org/Docs/Polynomials.pdf) comes out without having to calculate anything like 99^6 (a 12-digit number).

The Remainder and Factor theorems were surely known to Paolo Ruffini (1765–1822) who, modulo a few gaps, proved the impossibility of solving the quintic by radicals, and to William Horner (1786–1837); and probably well before that, to Déscartes, who indeed states the Factor theorem explicitly in his *La Géométrie* of 1637. Polish school students learn about the Factor Theorem under the name "twierdzenie Bézout" (Bézout Theorem) after Etienne Bézout (1730–1783) but this attribution is obscure.

Web link: eprints.soton.ac.uk/168861/. The Polish nomenclature is discussed here: pl.wikipedia.org/wiki/Twierdzenie_Bézouta (in Polish).

Further reading: *The Geometry of René Descartes*, 1925 annotated translation by David E. Smith and Marcia Latham, reprinted by Cosimo Classics, 2007 (in which copy the above citation is on p. 179).