Bernhard Riemann is 200

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Maths Study Group, LSBU
17th September, 2026

Bernhard Riemann
17/09/1826 – 20/07/1866

List of things named after Bernhard Riemann

Article Talk

From Wikipedia, the free encyclopedia

"Riemann" (by field) [edit]

- Riemann bilinear relations
- Riemann conditions
- Riemann form
- Riemann function
- Riemann–Hurwitz formula
- Riemann matrix
- Riemann operator
- Riemann singularity theorem
  - Riemann-Kempf singularity theorem
- Riemann surface
  - Compact Riemann surface
  - Planar Riemann surface
- Cauchy–Riemann manifold
  - The tangential Cauchy–Riemann complex
- Zariski–Riemann space

Less than 20 years' work!

Analysis [edit]

- Cauchy–Riemann equations
- Riemann integral
  - Generalized Riemann integral
  - Riemann multiple integral
- Riemann invariant
- Riemann mapping theorem
  - Measurable Riemann mapping theorem
- Riemann problem
- Riemann solver
- Riemann sphere

88 things named after Bernhard Riemann

Read Edit View history

From Wikipedia, the free encyclopedia

For music and Hugo Riemann, see Riemannian theory and Neo-Riemannian theory.

The German mathematician Bernhard Riemann (1826–1866) is the eponym of many things.

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- Riemann sphere
- Riemann–Hilbert correspondence
- Riemann–Hilbert problem
- Riemann–Lebesgue lemma
- Riemann–Liouville integral
- Riemann–Roch theorem
  - Arithmetic Riemann–Roch theorem
  - Riemann–Roch theorem for smooth manifolds
  - Riemann–Roch theorem for surfaces
  - Grothendieck–Hirzebruch–Riemann–Roch theorem
  - Hirzebruch–Riemann–Roch theorem
- Riemann–Stieltjes integral
- Riemann series theorem
- Riemann sum

Number theory [edit]

- Riemann–von Mangoldt formula
- Riemann hypothesis
  - Generalized Riemann hypothesis
  - Grand Riemann hypothesis
  - Riemann hypothesis for curves over finite fields
- Riemann theta function
- Riemann Xi function
- Riemann zeta function
- Riemann–Siegel formula
- Riemann–Siegel theta function

Physics [edit]

- Free Riemann gas also called primon gas
- Riemann invariant
- Riemann–Cartan geometry
- Riemann–Silberstein vector
- Riemann-Lebovitz formulation
- Riemann curvature tensor also called Riemann tensor
- Riemann tensor (general relativity)

Riemannian [edit]

- Pseudo-Riemannian manifold
- Riemannian bundle metric
- Riemannian circle
- Riemannian cobordism
- Riemannian connection
- Riemannian connection on a surface
- Riemannian cubic
- Riemannian cubic polynomials
- Riemannian foliation
- Riemannian geometry
  - Fundamental theorem of Riemannian geometry
- Riemannian graph
- Riemannian group
- Riemannian holonomy
- Riemannian manifold also called Riemannian space
- Riemannian metric tensor
- Riemannian Penrose inequality
- Riemannian polyhedron
- Riemannian singular value decomposition
- Riemannian submanifold
- Riemannian submersion
- Riemannian volume form
- Riemannian wavefield extrapolation
- Sub-Riemannian manifold
- Riemannian symmetric space

Riemann's [edit]

- Riemann's differential equation
- Riemann's existence theorem
- Riemann's explicit formula
- Riemann's minimal surface
- Riemann's theorem on removable singularities

Non-mathematical [edit]

- Free Riemann gas
- Riemann (crater)
- 4167 Riemann

Less than 20 years' work!!!!

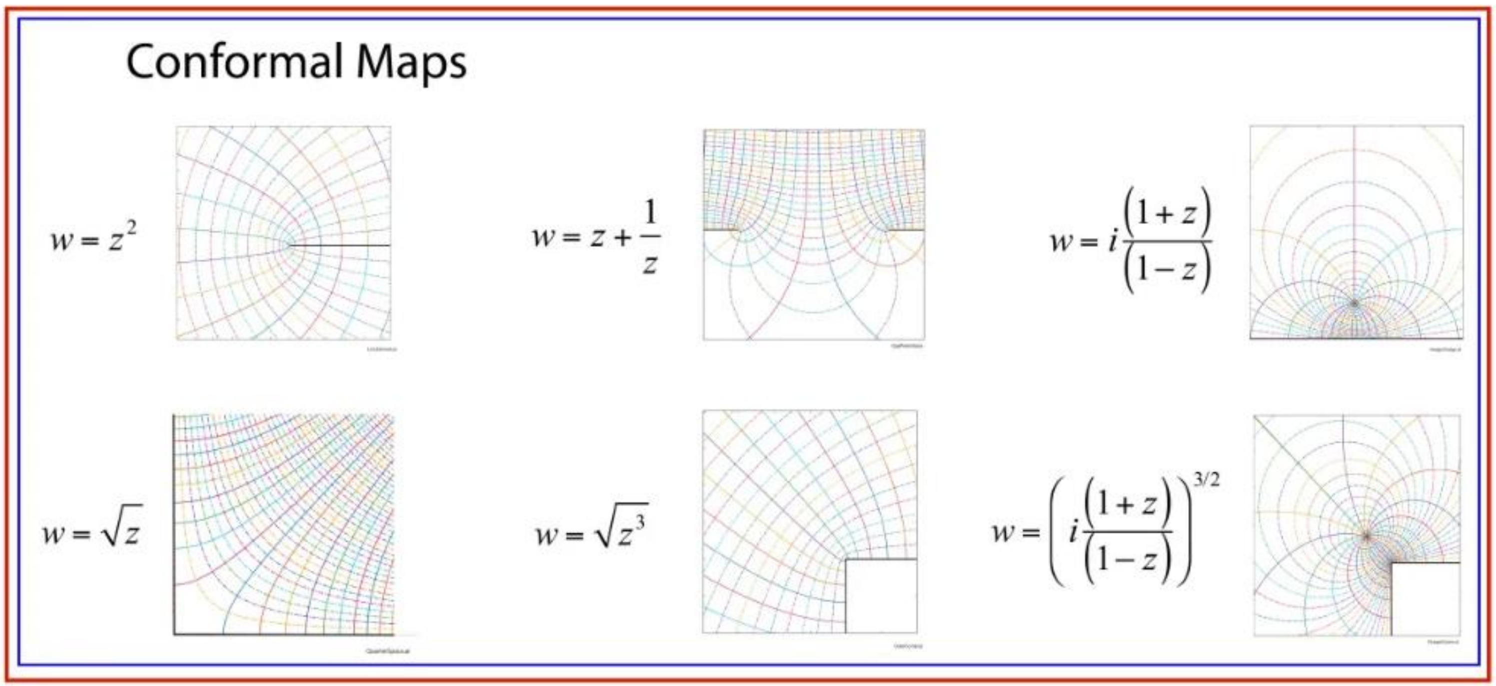
Early years

1826	Hannover	Born	2nd of 6 children of Lutheran minister
1840		Lyceum	
1842	Lüneburg	Gymnasium	Reads Legendre's *Number Theory* in 6 days
1846	Göttingen	University	Theology then philosophy (mathematics)
1847	Berlin	University	Steiner, Jacobi, **Dirichlet**, Eisenstein
1849	Göttingen	University	Gauss, Weber and Listing (physics)
1851			PhD (Gauss): geometric properties of analytic functions, **conformal mappings** and the connectivity of (Riemann) surfaces

Conformal maps

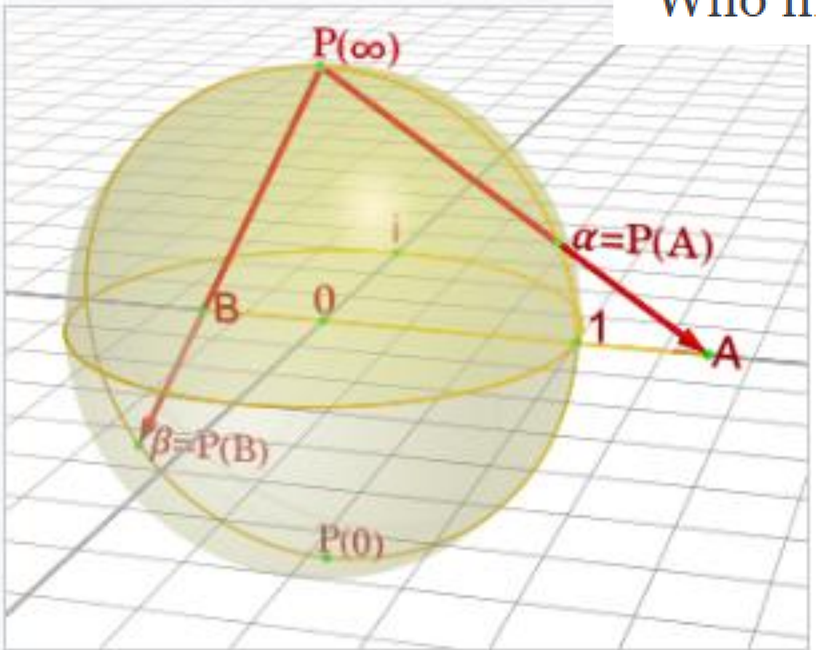
What is a conformal map? It is a transformation that takes one picture into another, keeping all local angles unchanged, no matter how distorted the overall transformation is.

This property was understood by the very first mathematicians, wrangling their sexagesimal numbers by the waters of Babylon as they mapped the heavens onto charts to foretell the coming of the seasons.



Source: galileo-unbound.blog/

Conformal maps: stereographic projection



The stereographic projection from the North pole of a sphere to its equatorial plane establishes a one to one correspondence between the sphere and the equatorial plane extended with a point at infinity denoted ∞. When the equatorial plane is the complex plane, this provides a visualization of the Riemann sphere

Source: Wikipedia

Who invented the Riemann Sphere?

8

After some searching I've found the following. Carl G. Neumann is given credit for inventing the Riemann Sphere. Also, one source gives Felix Klein independently -- Jeremy Gray in Worlds Out of Nothing: A course in the history of geometry in the 19th century, p. 344. Gray notes that "Klein observed [in a paper of 1874] that Riemann's treatment of what happened out towards infinity had the effect of making infinity a point, and that this could be seen by stereographic projection. We might add that, topologically, this is the one-point compactification of the plane." Gray goes on to say that Klein was uncertain what to make of this because the idea conflicted with what was currently accepted in projective geometry.

Now on to Neumann. Interestingly, the well known geometer H. S. M. Coxeter refers to the sphere as the Neumann Sphere on p. 258 of Section 14.6 in Non-Euclidean Geometry (1998) 6th edition. He might be referencing earlier books such as Theory of functions of a complex variable by Heinrich Burkhardt and Samuel Eugene Rasor (1913) which states on p. 47

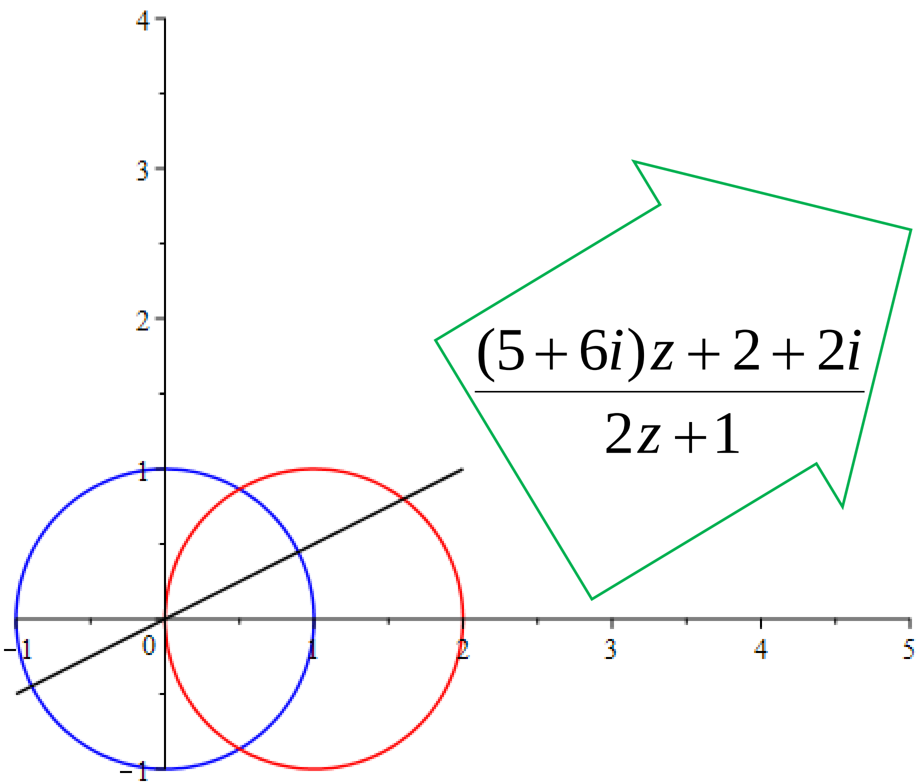
Source: math.stackexchange.com

Conformal maps: the Möbius transform

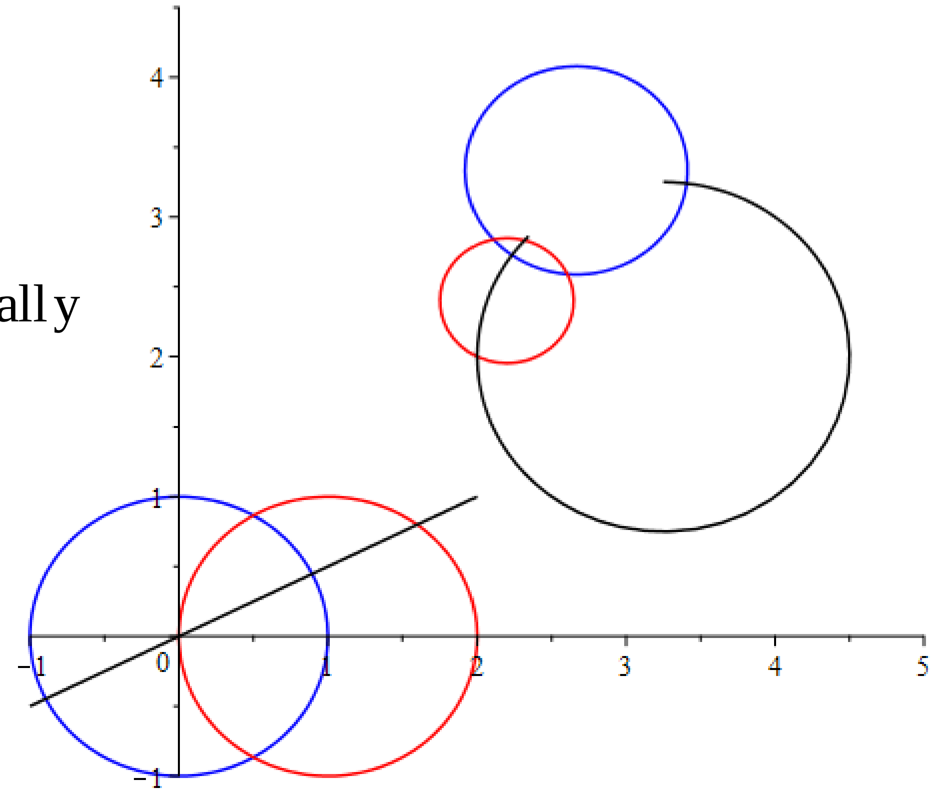
The mapping

$$f(z) = \frac{az+b}{cz+d}, \; ad - bc \neq 0$$

maps the complex plain to itself conformally



$$\frac{(5+6i)z+2+2i}{2z+1}$$



Note lines can map to circles and vice versa (line segments to arcs of circles).

The map here is a composition of a simpler transformation followed by a translation:

$$\frac{(5+6i)z+2+2i}{2z+1} = \frac{z-i}{2z+1} + 2z + 3$$

Properties of the Möbius transform

Simple Möbius transformations and composition [edit]

A Möbius transformation can be composed as a sequence of simple transformations.

The following simple transformations are also Möbius transformations:

- $f(z) = z + b \quad (a = 1, c = 0, d = 1)$ is a translation.
- $f(z) = az \quad (b = 0, c = 0, d = 1)$ is a combination of a homothety (uniform scaling) and a rotation. If $|a| = 1$ then it is a rotation, if $a \in \mathbb{R}$ then it is a homothety.
- $f(z) = 1/z \quad (a = 0, b = 1, c = 1, d = 0)$ (inversion and reflection with respect to the real axis)

Composition of simple transformations [edit]

If $c \neq 0$, let:

- $f_1(z) = z + d/c \quad$ (translation by *d*/*c*)
- $f_2(z) = \frac{1}{z} \quad$ (inversion and reflection with respect to the real axis)
- $f_3(z) = \frac{bc - ad}{c^2} z \quad$ (homothety and rotation)
- $f_4(z) = z + a/c \quad$ (translation by *a*/*c*)

Then these functions can be composed, showing that, if

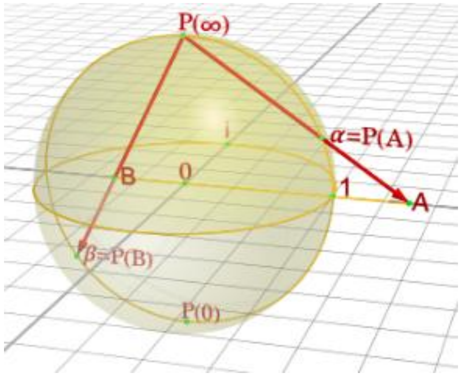
$$f(z) = \frac{az + b}{cz + d},$$

one has

$$f = f_4 \circ f_3 \circ f_2 \circ f_1.$$

Source: Wikipedia

Möbius and stereographic projection



Applying a Möbius transform to a point on the complex plain is the same as projecting that point on to the Riemann sphere, then rotating, translating, scaling and inverting the sphere. Then projecting back to the complex plain

Mature years

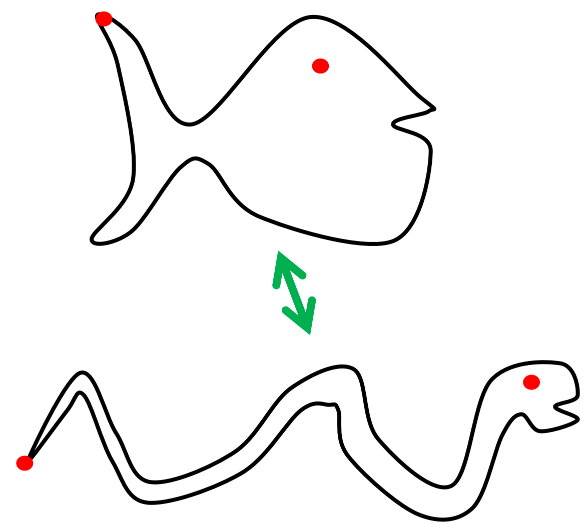
1851	PhD under Gauss. Dissertation: *Foundations of a general theory of functions of a variable complex magnitude* Appointed to teaching post at Göttingen. Habilitation work (30 months)
1854	Habilitation dissertation: representation of functions by trig series, Riemann integral. Habilitation lecture: *On the hypotheses at the foundations of geometry*: Riemannian space, curvature tensor, physical theory of geometry (precursor to modern cosmology)
1855	Death of Gauss whose chair is taken by Dirichlet.
1857	Personal chair at Göttingen. Classic: *Theory of abelian functions* (developed for a course given to 3 students, incl. Dedekind)
1858	Visit from Italian school of geometry (Betti, Casorati, Brioschi)
1859	Death of Dirichlet: Riemann accedes to the chair of mathematics. Elected to Berlin Academy of Sciences. Report to Academy: *On the number of primes less than a given magnitude*, Explicit Formula and Hypothesis

Riemann's 1951 dissertation presented a version of what became the Riemann Mapping Theorem , according to Lars Ahlfors, "one of the most important theorems of complex analysis".

The Riemann Mapping Theorem

In a translation of Riemann’s original assertion:

> Two given simply connected plane surfaces can always be mapped onto one another in such a way that each point of the one corresponds to a unique point of the other in a continuous way and the correspondence is conformal; moreover, the correspondence between an arbitrary interior point and an arbitrary boundary point of the one and the other may be given arbitrarily, but when this is done the correspondence is determined completely.



Sixty years’ work was required to make this entirely rigorous! Ahlfors, writing in 1953, commented that Riemann sends “almost cyptic messages to the future” and that his statement of the mapping theorem “would defy any attempt at proof, even with modern methods”

English translation: www.scribd.com/document/478123440/1408-0132v1

History: Jeremy Gray, "On the history of the Riemann mapping theorem" (PDF), *Rendiconti del Circolo Matematico di Palermo. Serie II.* Supplemento (34): 1994, pp. 47–94.

The Riemann Mapping Theorem: modern day

Riemann mapping theorem

Article Talk

From Wikipedia, the free encyclopedia

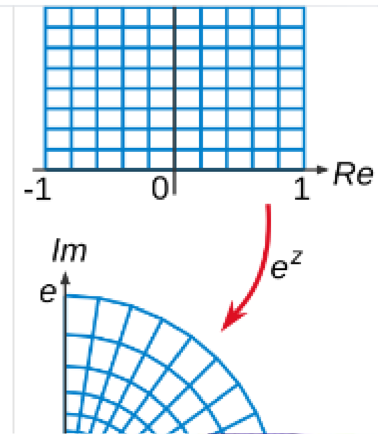
In complex analysis, the **Riemann mapping theorem** states that if U is a non-empty simply connected open subset of the complex number plane $\mathbb{C}$ which is not all of $\mathbb{C}$, then there exists a biholomorphic mapping f (i.e. a bijective holomorphic mapping whose inverse is also holomorphic) from U onto the open unit disk

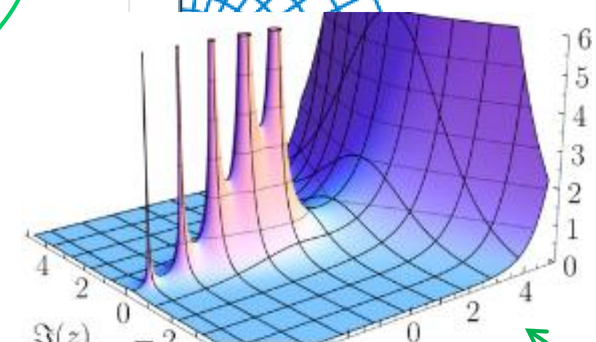
$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

This mapping is sometimes called the **Riemann mapping** from U to the unit disk.[1]

Intuitively, the condition that U be simply connected means that U does not contain any "holes". The fact that f is biholomorphic implies that it is a conformal map and therefore angle-preserving. Such a map may be interpreted as preserving the shape of any sufficiently small figure, while possibly rotating and scaling (but not reflecting) it.

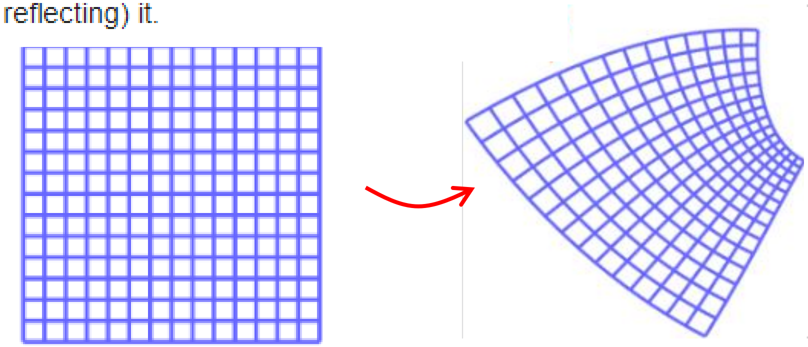
In the mathematical theory of functions of one or more complex variables, and also in complex algebraic geometry, a **biholomorphism** or **biholomorphic function** is a bijective holomorphic function whose inverse is also holomorphic. Note that the las





In mathematics, a **holomorphic function** is a complex-valued function of one or more complex variables that is complex differentiable in a neighbourhood of each point in a domain in complex coordinate space $\mathbb{C}^n$. The existence of a complex derivative in a neighbourhood is a very strong conditio

the Gamma function



A rectangular grid (top) and its image under a conformal map f (bottom). It is seen that f maps pairs of lines intersecting at 90° to pairs of curves still intersecting at 90°.

The Riemann Mapping Theorem: modern day

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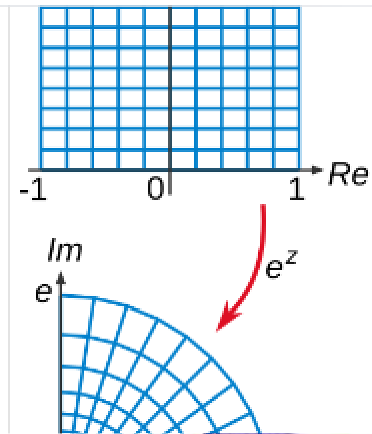

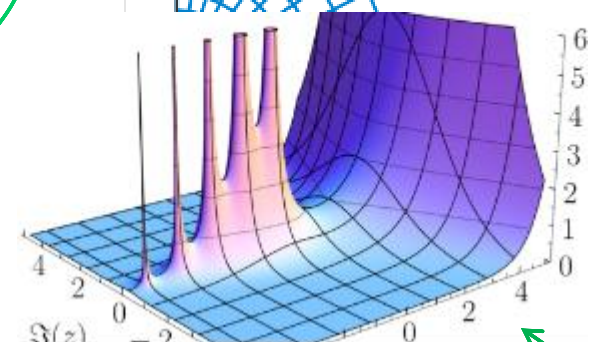

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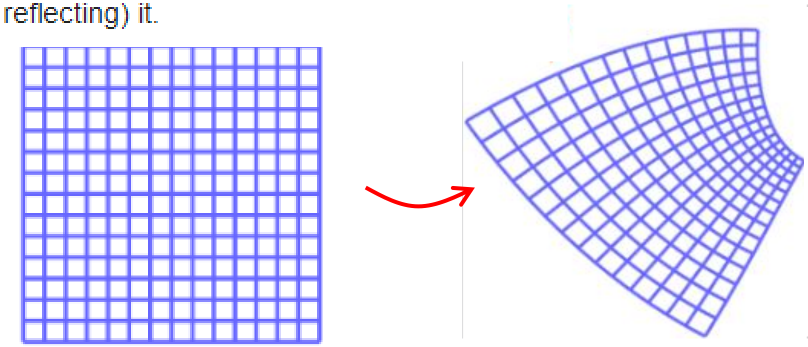


A rectangular grid (top) and its image under a conformal map f (bottom). It is seen that f maps pairs of lines intersecting at 90° to pairs of curves still intersecting at 90°.

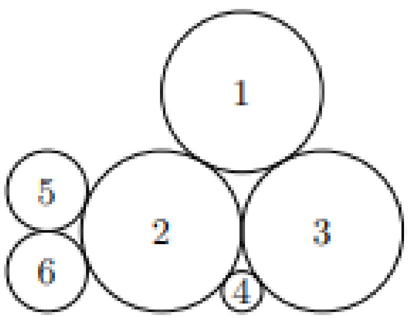
The Mapping Theorem and circle packings (1)

The Riemann Mapping Theorem says that any proper open subset of the complex plane may be mapped conformally on to the open unit disk at the origin.

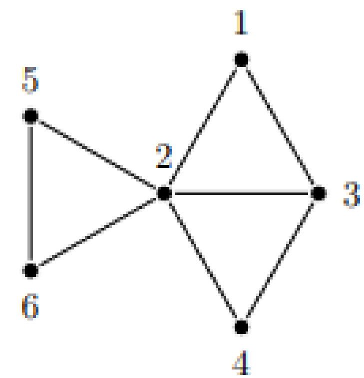
This mapping is unique up to Möbius transformations of the disk.

This uniqueness led William Thurston to draw a striking comparison with **circle packings**.

A circle packing is a collection of circles in the complex plane which are disjoint except possibly on their boundaries – there are allowed to be tangencies.



A planar graph may be drawn, called the **nerve** of the packing, with adjacencies corresponding to tangencies



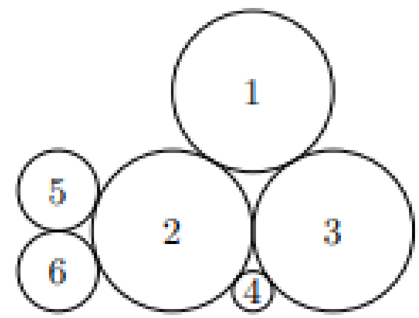
Images: Asaf Nachmias, "Planar maps, random walks and circle packing", Lecture notes of the 48th Saint-Flour summer school, Tel Aviv University, 2018.

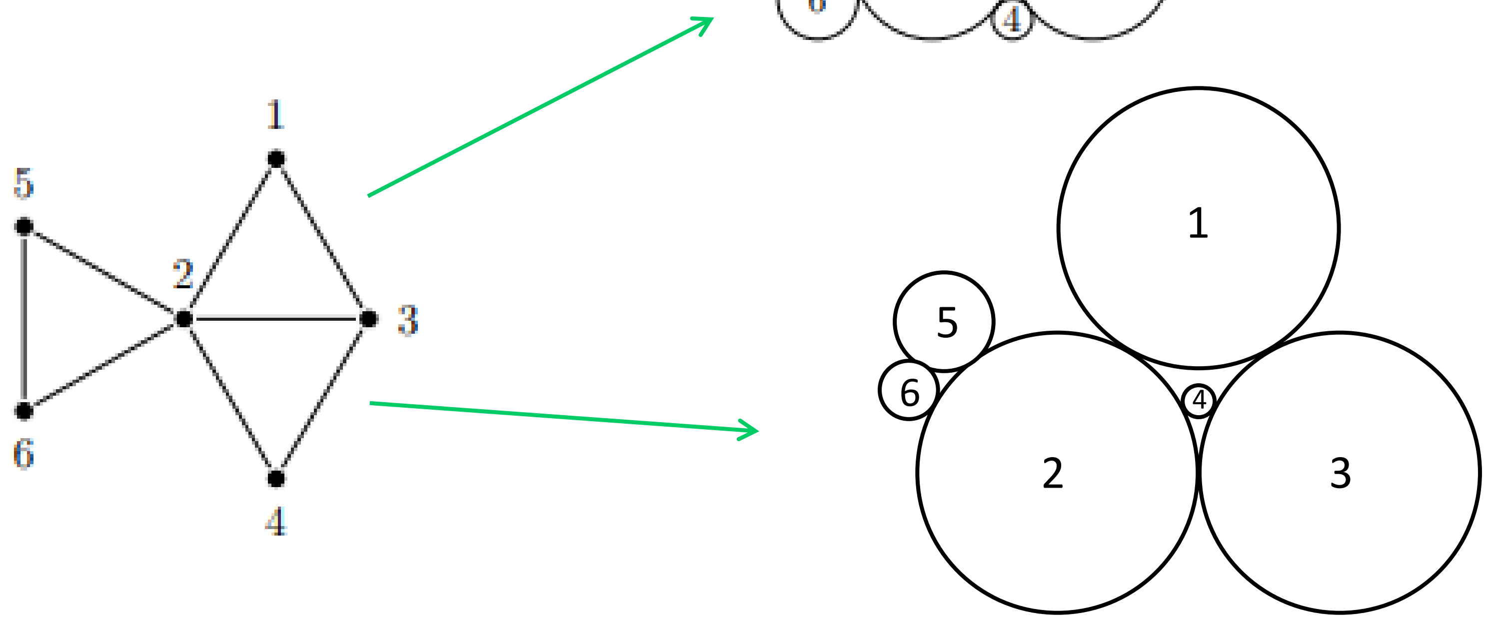
The Mapping Theorem and circle packings (2)

Let *P* be a circle packing and denote by *G(P)* its nerve.

Circle Packing Theorem Every finite simple planar graph *G* has a circle packing. That is, there exists a circle packing *P* such that *G(P)* is isomorphic to *G*.

The circle packing need not be unique.





The Mapping Theorem and circle packings (3)

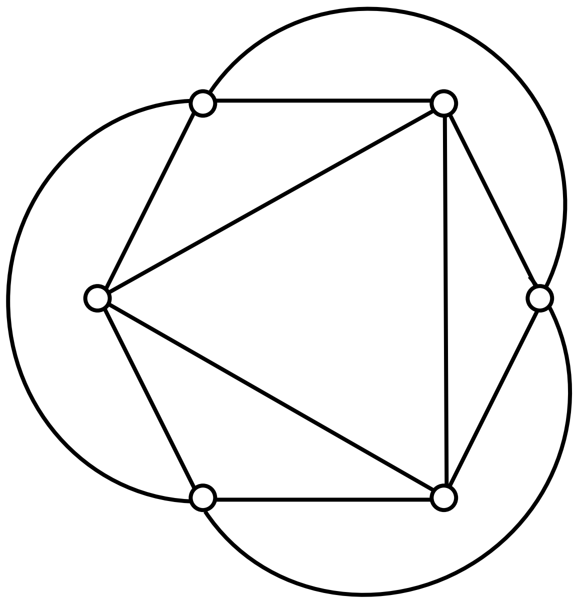
However if a graph is maximally planar then its circle packing is unique up to Möbius transformations.

Unique Circle Packing Theorem (Koebe–Andreev–Thurston) If a connected planar graph is maximal (i.e., no further edge can be added to it without destroying planarity), then the circle packing given by the above theorem is unique up to reflections and Möbius transformations.

Exercise 5 Let G be a connected planar graph with $n \geq 3$ vertices. Show that the following are equivalent:

- (i) G is a maximal planar graph.
- (ii) G has $3n-6$ edges.
- (iii) Every drawing D of G divides the plane into faces that have three edges each, and each edge is adjacent to two distinct faces. (This includes one unbounded face.)
- (iv) At least one drawing D of G divides the plane into faces that have three edges each, and each edge is adjacent to two distinct faces.

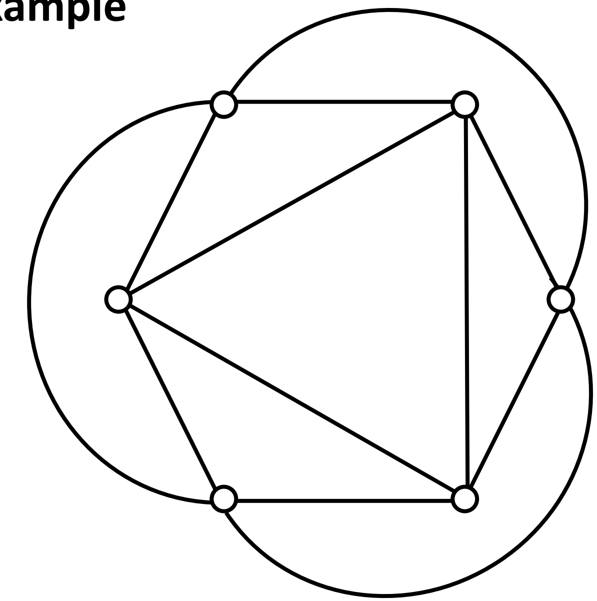
(*Hint:* you may use without proof Euler's formula $V - E + F = 2$ for planar graphs, where F is the number of faces including the unbounded face.)



Source: Terence Tao, 246C notes 2: Circle packings, conformal maps, and quasiconformal maps terrytao.wordpress.com/2018/04/12/246c-notes-2-circle-packings-conformal-maps-and-quasiconformal-maps/

The Mapping Theorem and circle packings (4)

Example



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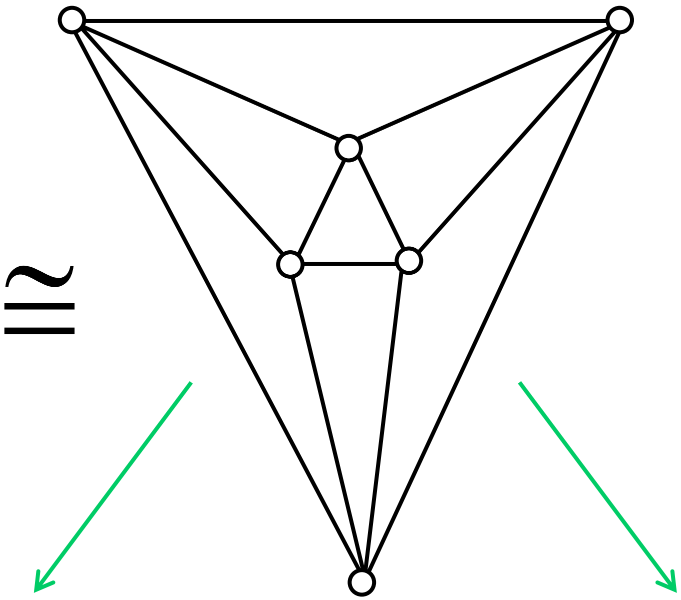
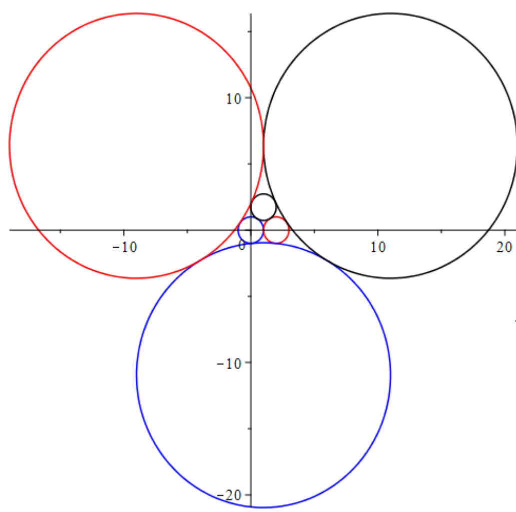
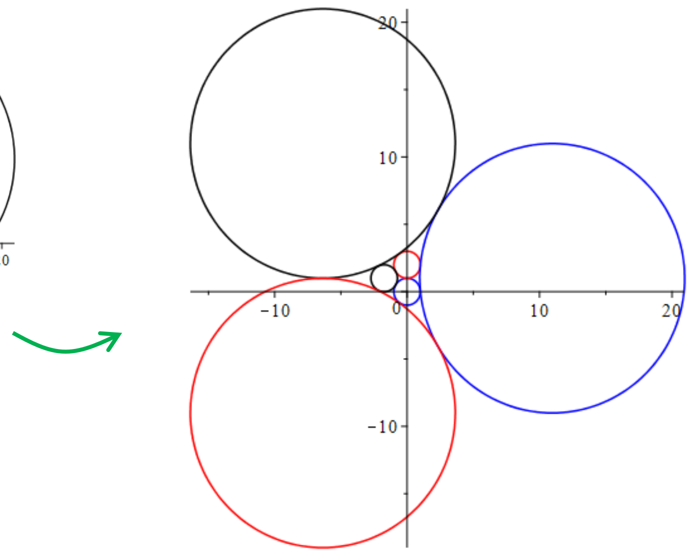






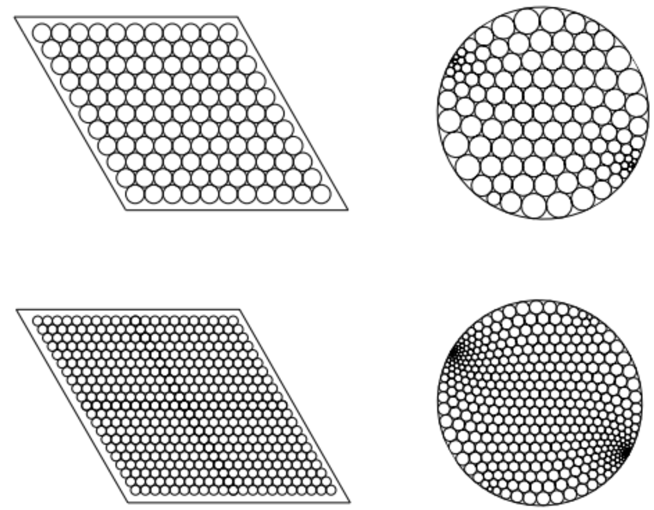
The Mapping Theorem and circle packings (5)

Thurston conjectured that circle packings can be used to approximate the conformal map arising in the Riemann mapping theorem. Here is an informal statement:

Conjecture 6 (Informal Thurston conjecture) Let U be a simply connected domain, with two distinct points z_0, z_1. Let $\phi : U \to D(0,1)$ be the conformal map from U to $D(0,1)$ that maps z_0 to the origin and z_1 to a positive real. For any small $\varepsilon > 0$, let C_ε be the portion of the regular hexagonal circle packing by circles of radius ε that are contained in U, and let C'_ε be an circle packing of $D(0,1)$ with the same nerve (up to isomorphism) as C_ε, with all "boundary circles" tangent to $D(0,1)$, giving rise to an "approximate map" $\phi_\varepsilon : U_\varepsilon \to D(0,1)$ defined on the subset U_ε of U consisting of the circles of C_ε, their interiors, and the interstitial regions between triples of mutually tangent circles. Normalise this map so that $\phi_\varepsilon(z_0)$ is zero and $\phi_\varepsilon(z_1)$ is a positive real. Then ϕ_ε converges to ϕ as $\varepsilon \to 0$.

A rigorous version of this conjecture was proven by Rodin and Sullivan. Besides

Source: Terence Tao, 246C notes 2: Circle packings, conformal maps, and quasiconformal maps
terrytao.wordpress.com/2018/04/12/246c-notes-2-circle-packings-conformal-maps-and-quasiconformal-maps/

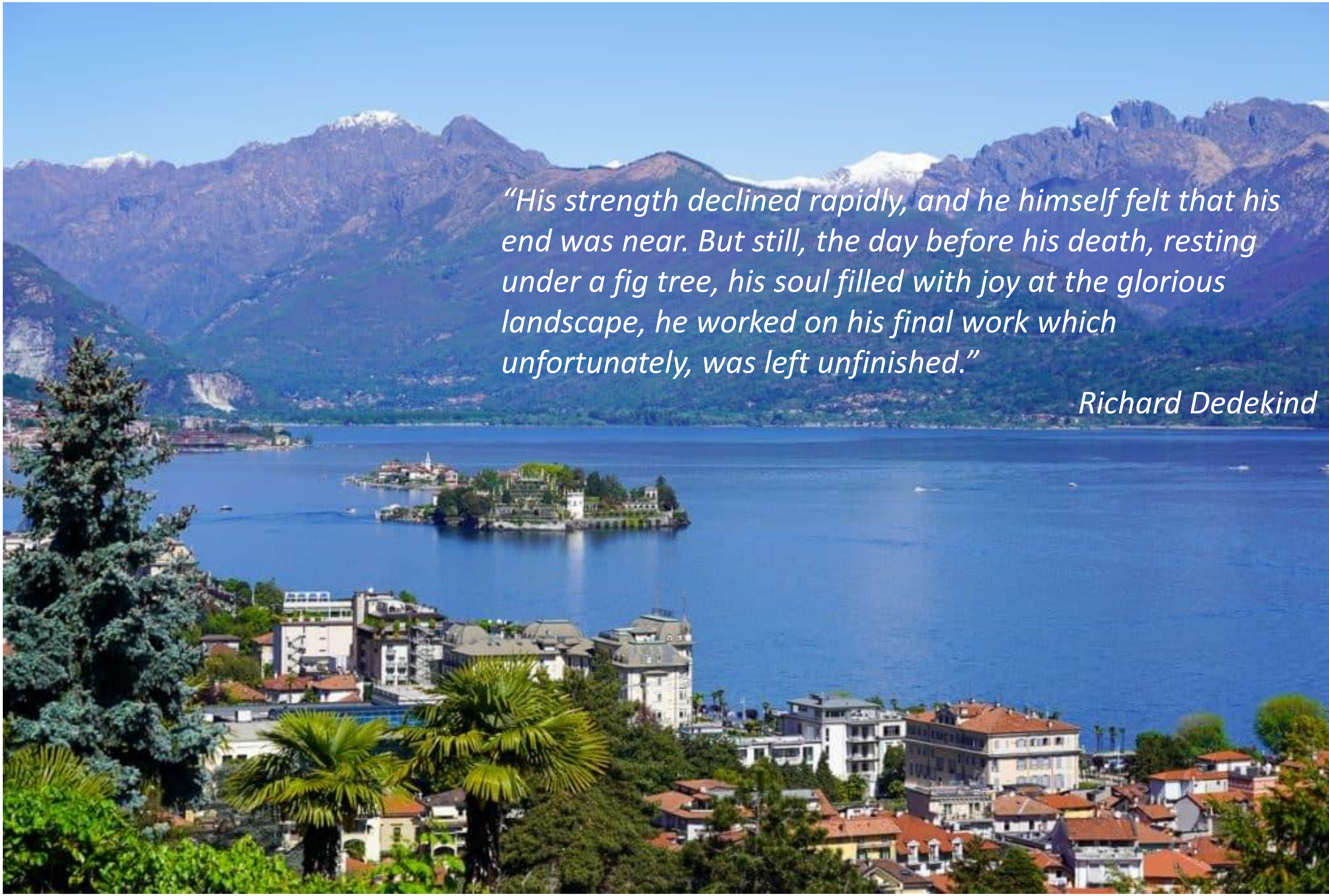


Approximating the conformal map from a rhombus to the disk using circle packing, at two different degrees of accuracy.

Source: Asaf Nachmias, "Planar maps, random walks and circle packing", Lecture notes of the 48th Saint-Flour summer school, Tel Aviv University, 2018.

Last years

1862	Marries (one daughter). Contracts tuberculosis. Travels in Italy
1860s	Ill health. Travels between Göttingen and Italy.
1866	Dies 20 July 1866 (aged 39) in Selasca, a hamlet on the western shores of Lake Maggiore) in Northen Italy


"His strength declined rapidly, and he himself felt that his end was near. But still, the day before his death, resting under a fig tree, his soul filled with joy at the glorious landscape, he worked on his final work which unfortunately, was left unfinished."
Richard Dedekind