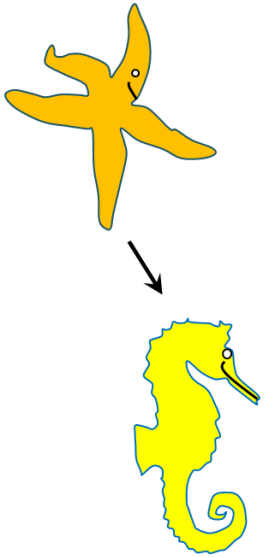
THEOREM OF THE DAY

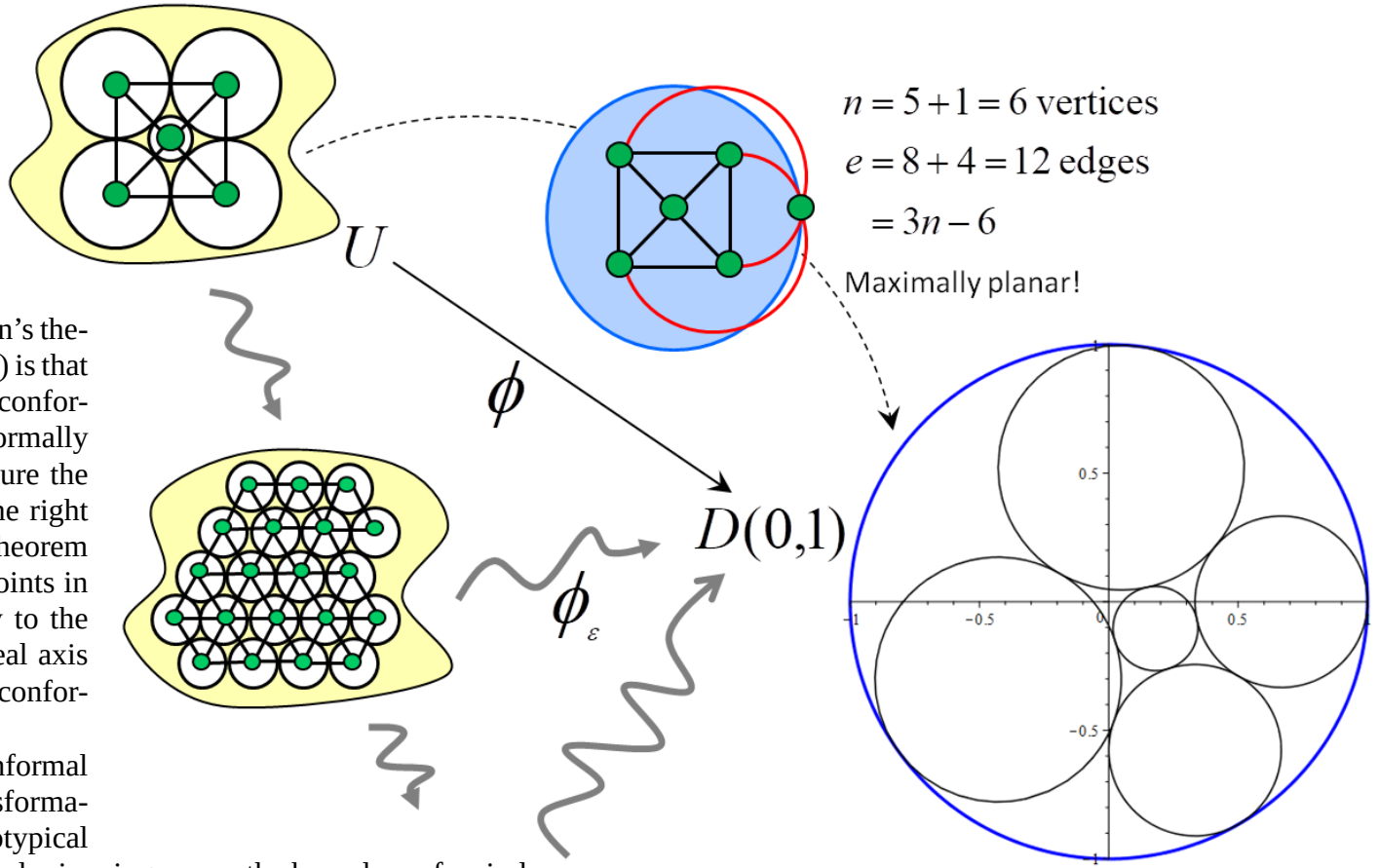
The Riemann Mapping Theorem *Let U be a simply connected open subset of $\mathbb{C}$ that is not all of $\mathbb{C}$. Then there is a conformal mapping from U the open unit disk $D(0,1)$.*

Informally, we may say that U is a region of the complex plane which has no holes (simply connected) and is taken without its boundary (open). A mapping is conformal on an open set if it is bijective and holomorphic (complex differentiable at every point in the set). Conformal mappings preserve some essential geometric properties such as angles and orientations: locally, things preserve their shapes although not their sizes.



An immediate consequence of Riemann's theorem (and how he originally asserted it) is that any two simply connected regions are conformally equivalent: a star fish is conformally equivalent to a sea horse. In our picture the mapping has even kept the faces in the right place! This is because Riemann's theorem gives us the freedom to choose two points in our set U and map them respectively to the origin and a point of $D(0,1)$ on the real axis of the complex plane. After that his conformal mapping is uniquely determined.

Without fixing points, Riemann's conformal mapping is unique up to Möbius transformations of the unit disk. These are prototypical conformal mappings built by translation, scaling, rotation and mirroring across the boundary of a circle.



In 1985, the geometer William Thurston was inspired by a seemingly quite unrelated instance of uniqueness up to Möbius transformations. Any maximally planar graph corresponds to a circle packing in the complex plane. An example is shown above: the edges of the graph correspond to tangencies among the circles. This circle packing also is unique up to Möbius transformations (for instance, an alternative packing matching our planar graph can be drawn by mirroring the circles inside the unit disk across to outside its boundary). Thurston conjectured that Riemann's conformal mapping could be explicitly constructed by packing the set U with little circles and mapping these, and the spaces between them, to a circle packing inside $D(0,1)$ (as in our illustration). Packing with ever more and smaller circles should approximate Riemann's mapping ever more closely: in our picture, suppose the circles have radius ε, then $\phi_\varepsilon \to \phi$ as $\varepsilon \to 0$. And so it was quickly proved to be!

Leading 20th analyst Lars Ahlfors called this "one of the most important theorems of complex analysis"; not bad for a 25-year-old doctoral student! In fact, Riemann's 1851 thesis was far ahead of its time. Ahlfors complains admiringly that Riemann was sending "almost cryptic messages to the future"; it would be 60 years before his ideas were made fully rigorous.

Web link: terrytao.wordpress.com/2018/04/12/

Further reading: *Introduction to Circle Packing: The Theory of Discrete Analytic Functions* by Kenneth Stephenson, C.U.P., 2005.