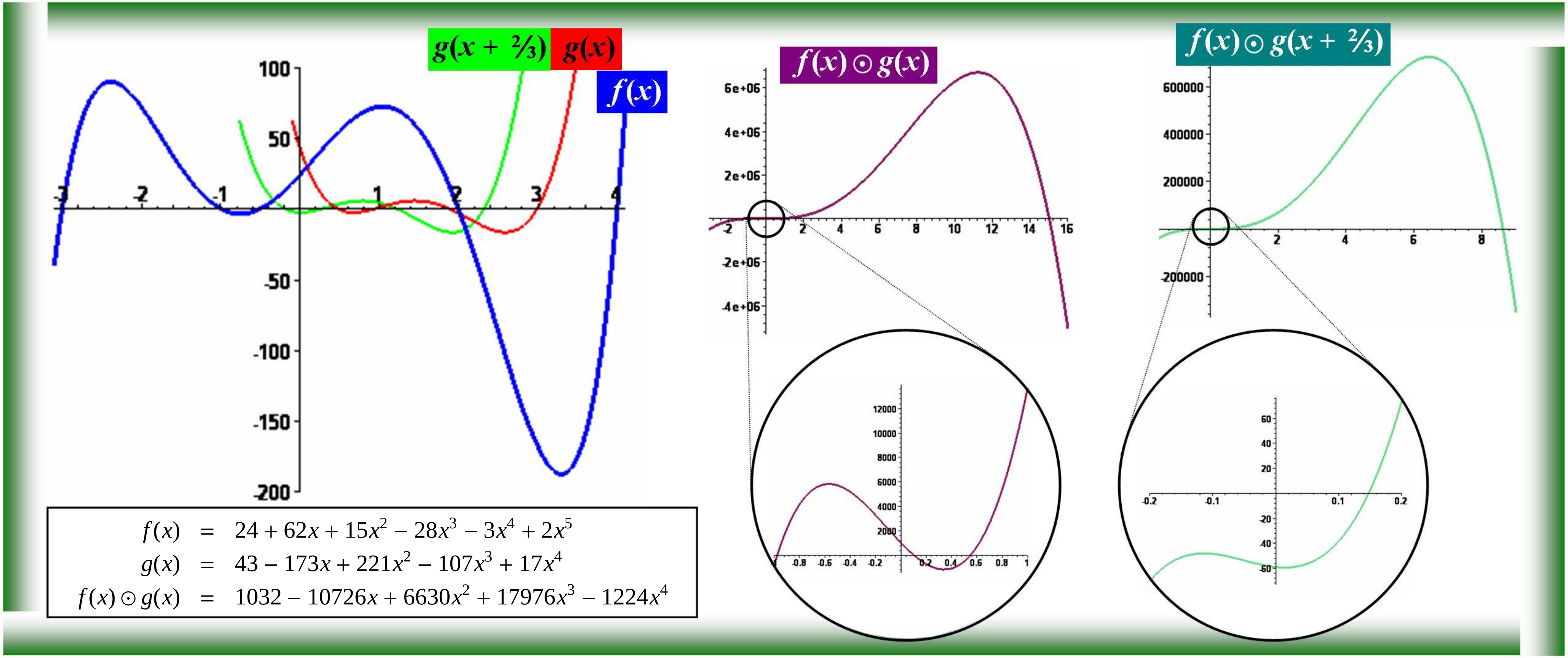
THEOREM OF THE DAY

A Theorem of Schur on Real-Rootedness *Let $f(x) = a_0 + a_1x + \ldots + a_mx^m$ and $g(x) = b_0 + b_1x + \ldots + b_nx^n$ be polynomials in $\mathbb{R}[x]$ and suppose that f and g have only real zeros and that all the zeros of g have the same sign. Then the polynomial $f(x) \odot g(x) = \sum_{k=0}^{\min(m,n)} k!a_kb_kx^k$ has only real zeros and if a_0 and b_0 are both nonzero then these zeros are distinct.*



$$
\begin{aligned}
f(x) &= 24 + 62x + 15x^2 - 28x^3 - 3x^4 + 2x^5\\
g(x) &= 43 - 173x + 221x^2 - 107x^3 + 17x^4\\
f(x) \odot g(x) &= 1032 - 10726x + 6630x^2 + 17976x^3 - 1224x^4
\end{aligned}
$$

Two functions, f and g, satisfying the conditions of this theorem, are plotted above left. If g is shifted left by the transformation $x \mapsto x + 2/3$ then g still has only real zeros but these take both positive and negative signs, so the theorem need not apply; and indeed, although $f(x) \odot g(x)$ and $f(x) \odot g(x + 2/3)$ look the same on the large scale (above, top right), zooming in (above, bottom right) shows that the latter only has one real zero near the origin.

Omitting from $\odot$ the $k!$ multiplier, leaving $\sum a_kb_kx^k$, reduces this theorem to an 1895 theorem of Ernest Malo. Issai Schur's 1914 result is stronger by virtue of an earlier result of Edmond Laguerre: if $\sum c_kx^k$ has real roots then so does $\sum (c_k/k!)x^k$.

Web link: www.math.hawaii.edu/~tom/mathfiles/czdssurvey.pdf

Further reading: *Problems and Theorems in Analysis, vol. 2* by George Pólya and Gabor Szegő, Springer-Verlag Berlin, 1998, part V, ch. 2, §3.