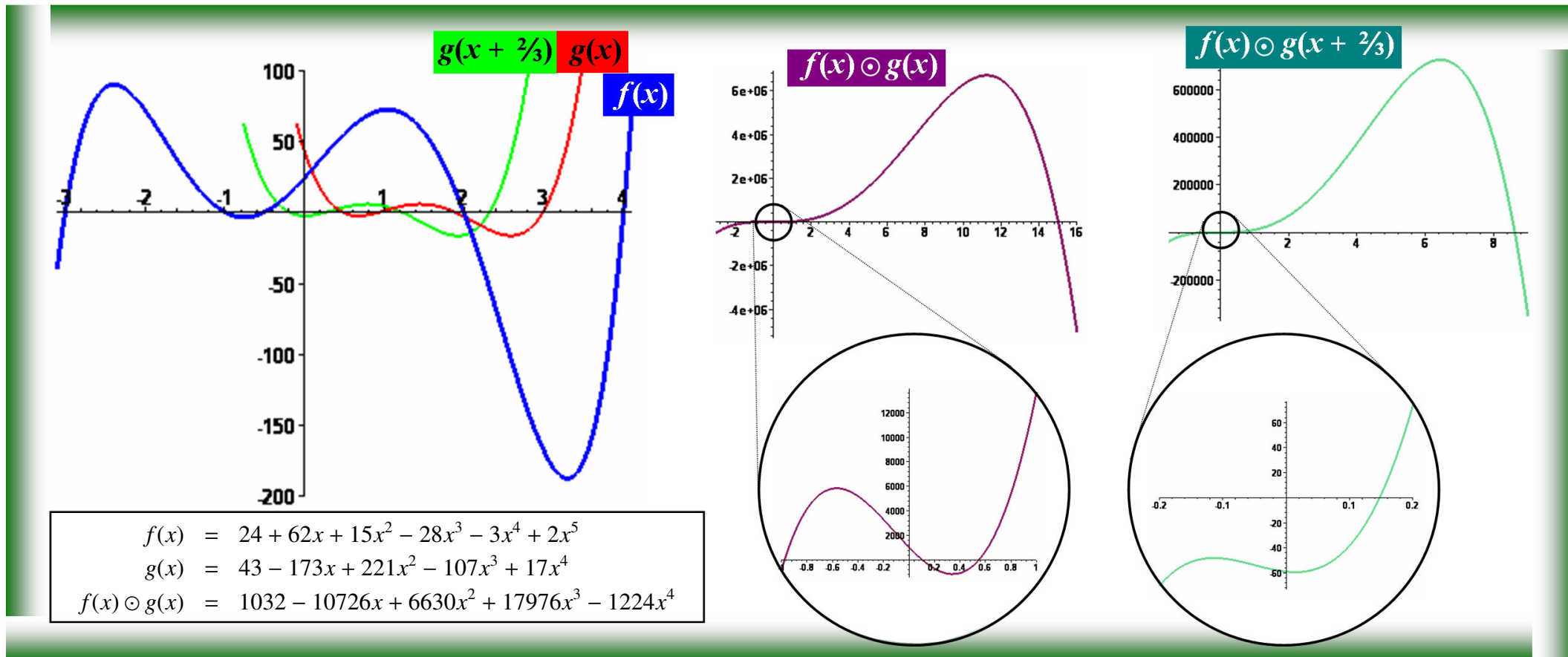




THEOREM OF THE DAY

A Theorem of Schur on Real-Rootedness Let $f(x) = a_0 + a_1x + \dots + a_mx^m$ and $g(x) = b_0 + b_1x + \dots + b_nx^n$ be polynomials in $\mathbb{R}[x]$ and suppose that f and g have only real zeros and that all the zeros of g have the same sign. Then the polynomial $f(x) \odot g(x) = \sum_{k=0}^{\min(m,n)} k! a_k b_k x^k$ has only real zeros and if a_0 and b_0 are both nonzero then these zeros are distinct.



Two functions, f and g , satisfying the conditions of this theorem, are plotted above left. If g is shifted left by the transformation $x \mapsto x + 2/3$ then g still has only real zeros but these take both positive and negative signs, so the theorem need not apply; and indeed, although $f(x) \odot g(x)$ and $f(x) \odot g(x + 2/3)$ look the same on the large scale (above, top right), zooming in (above, bottom right) shows that the latter only has one real zero near the origin.

Omitting from \odot the $k!$ multiplier, leaving $\sum a_k b_k x^k$, reduces this theorem to an 1895 theorem of Ernest Malo. Issai Schur's 1914 result is stronger by virtue of an earlier result of Edmond Laguerre: if $\sum c_k x^k$ has real roots then so does $\sum (c_k/k!) x^k$.

Web link: www.math.hawaii.edu/~tom/mathfiles/czdssurvey.pdf

Further reading: *Problems and Theorems in Analysis, vol. 2* by George Pólya and Gabor Szegő, Springer-Verlag Berlin, 1998, part V, ch. 2, §3.

