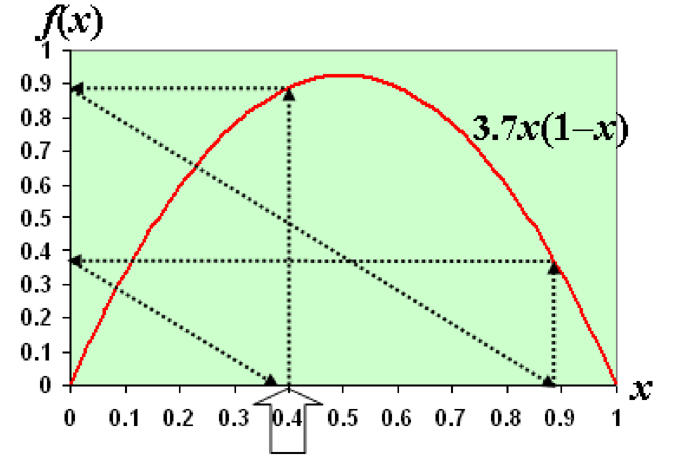
THEOREM OF THE DAY

Sharkovsky's Theorem *Specify an ordering, ≺, of the positive integers:*

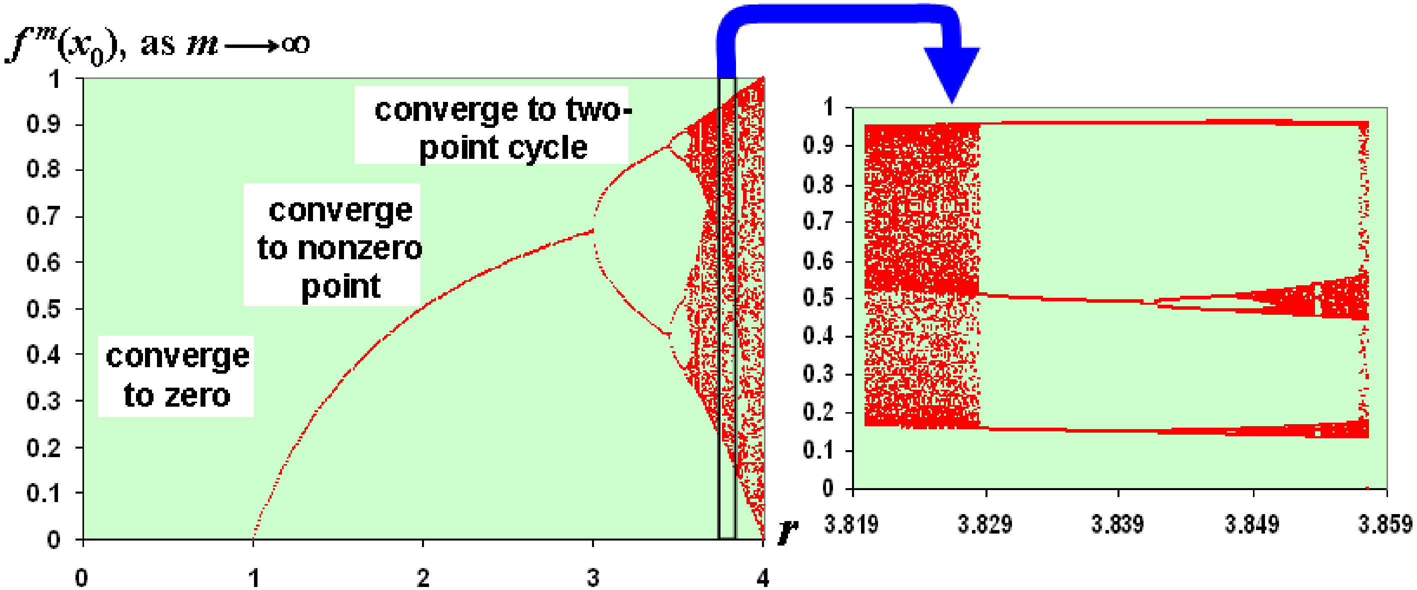
$$3, 5, 7, 9, \ldots, 2\times3, 2\times5, 2\times7, 2\times9, \ldots, 2^2\times3, 2^2\times5, 2^2\times7, 2^2\times9, \ldots \quad \ldots, 2^4, 2^3, 2^2, 2^1, 1,$$

defined formally as follows: take $x < y$ with x and y written (uniquely) as $x = 2^r p$ and $y = 2^s q$, p, q odd; then $x \prec y$ if $r \leq s$ and $p > 1$; otherwise $y \prec x$. Now let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function having a point x of period m; that is, $f^m(x) = f(x)$, where f^m denotes the m-th iteration of f. Then for every n with $m \prec n$, f has some point of period n. In particular, if f has a point of period 3, then f has periods of all positive integer orders.



Iterating the logistic map $f(x) = rx(1-x)$ from $x_0 = 0.4$, using $r = 3.7$:

$0.4 \to 0.89 \to 0.36 \to 0.85 \to \ldots$



Bifurcation diagrams showing long-term behaviour of the iterated logistic map for different r.

For any choice of parameter r, the *logistic map*, $f(x) = rx(1-x)$, certainly gives a continuous curve, as is illustrated in the left-hand plot (using $r = 3.7$). The long-term behaviour of the iterated logistic map is displayed in the right-hand plots: choose an arbitrary initial value $x_0 \in (0, 1)$ and calculate the sequence $f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots$; for values of r between zero and $r \approx 3.57$ the sequence leads our arbitrary initial value to convergence in a unique cycle through 2^i values, $i = 0, 1, 2, \ldots$ (period 2^i). Beyond 3.57, chaos ensues: indeed, the choice $r = 3.7$ appears to provide no convergent behaviour, and a tiny change to x_0 eventually causes a large change to our sequence. Suddenly, around $r = 3.83$, a window of order opens! On the far right, magnification shows convergence to a cycle of period three. But here Sharkovsky's theorem predicts periods of *any* length. Thus some choice of x_0 will converge to a cycle of length, say, 100; This cycle will not be an *attractor*: an infinitesimal change to x_0 will drive us back to period 3. Such non-attracting periods are invisible to computers!

The Ukranian O.M. Sharkovsky's 1964 theorem shows that another world exists behind the complexity of fractal plots.

Web link: www.mcasco.com/Order/ordintro.html: an excellent free on-line course by J.D. Jones for M. Casco Associates.

Further reading: *Over and Over Again* by Gengzhe Chang and Thomas W. Sederberg, Mathematical Association of America, 1998, chapter 23.