Sharkovsky’s Theorem Specify an ordering, <, of the positive integers:

\[ 3, 5, 7, 9, \ldots, 2 \times 3, 2 \times 5, 2 \times 7, 2 \times 9, \ldots, 2^2 \times 3, 2^2 \times 5, 2^2 \times 7, 2^2 \times 9, \ldots, 2^4, 2^3, 2^2, 2^1, 1, \]
defined formally as follows: take \( x < y \) with \( x \) and \( y \) written (uniquely) as \( x = 2^p r \) and \( y = 2^s q \), \( p, q \) odd; then \( x < y \) if \( r \leq s \) and \( p > 1 \); otherwise \( y < x \). Now let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function having a point \( x \) of period \( m \); that is, \( f^m(x) = f(x) \), where \( f^m \) denotes the \( m \)-th iteration of \( f \). Then for every \( n \) with \( m < n \), \( f \) has some point of period \( n \). In particular, if \( f \) has a point of period 3, then \( f \) has periods of all positive integer orders.

For any choice of parameter \( r \), the logistic map, \( f(x) = rx(1-x) \), certainly gives a continuous curve, as is illustrated in the left-hand plot (using \( r = 3.7 \)).

Iterating the logistic map \( f(x) = rx(1-x) \) from \( x_0 = 0.4 \), using \( r = 3.7 \):

\[ 0.4 \rightarrow 0.89 \rightarrow 0.36 \rightarrow 0.85 \rightarrow \ldots \]

Bifurcation diagrams showing long-term behaviour of the iterated logistic map for different \( r \). For any choice of parameter \( r \), the logistic map, \( f(x) = rx(1-x) \), certainly gives a continuous curve, as is illustrated in the left-hand plot (using \( r = 3.7 \)).

Beyond 3.57, chaos ensues: indeed, the choice \( r = 3.7 \) appears to provide no convergent behaviour, and a tiny change to \( x_0 \) eventually causes a large change to our sequence. Suddenly, around \( r = 3.83 \), a window of order opens! On the far right, magnification shows convergence to a cycle of period three. But here Sharkovsky’s theorem predicts periods of any length. Thus some choice of \( x_0 \) will converge to a cycle of length, say, 100; This cycle will not be an attractor: an infinitesimal change to \( x_0 \) will drive us back to period 3. Such non-attracting periods are invisible to computers!

The Ukrainian O.M. Sharkovsky’s 1964 theorem shows that another world exists behind the complexity of fractal plots.


Further reading: *Over and Over Again* by Gengzhe Chang and Thomas W. Sederberg, Mathematical Association of America, 1998, chapter 23.