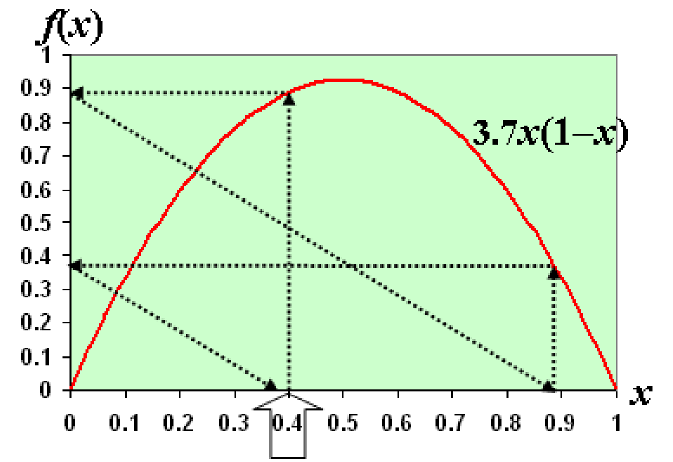
THEOREM OF THE DAY

Sharkovsky's Theorem *Specify an ordering,* $\prec$*, of the positive integers:*

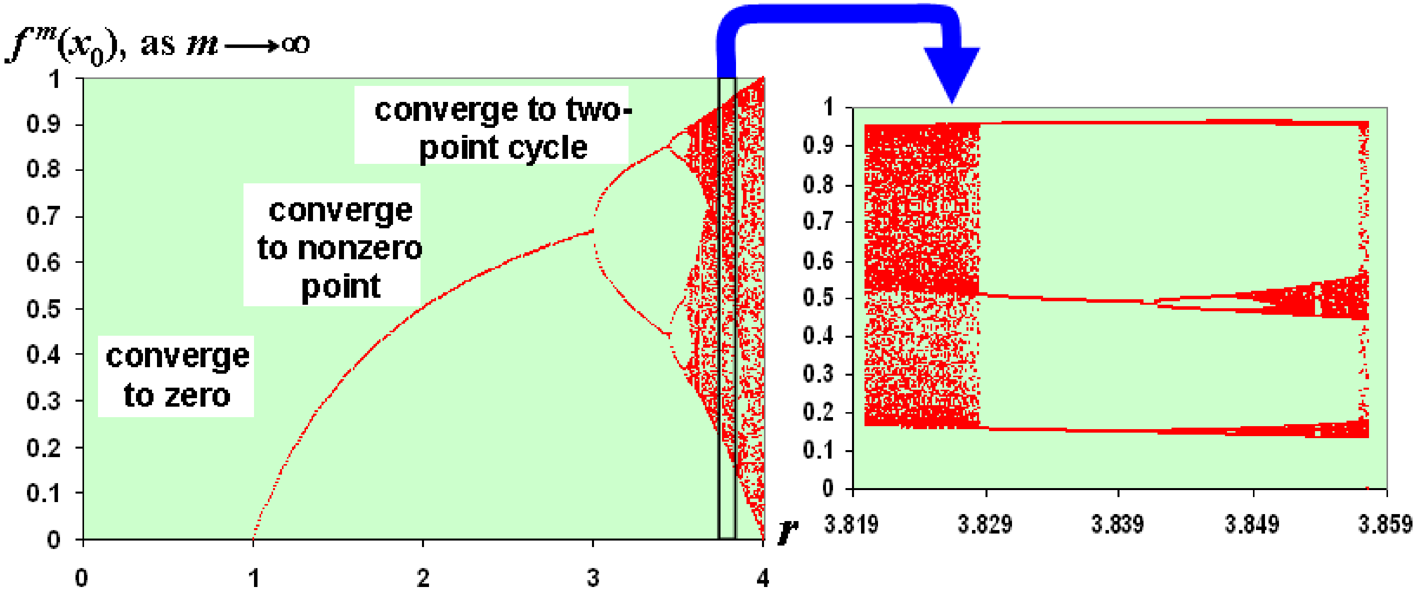
$$3, 5, 7, 9, \ldots, 2\times 3, 2\times 5, 2\times 7, 2\times 9, \ldots, 2^2\times 3, 2^2\times 5, 2^2\times 7, 2^2\times 9, \ldots \quad \ldots, 2^4, 2^3, 2^2, 2^1, 1,$$

defined formally as follows: take $x < y$ *with* x *and* y *written (uniquely) as* $x = 2^r p$ *and* $y = 2^s q$*,* p*,* q *odd; then* $x \prec y$ *if* $r \leq s$ *and* $p > 1$*; otherwise* $y \prec x$*. Now let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function having a point* x *of period* m*; that is,* $f^m(x) = f(x)$*, where* f^m *denotes the* m*-th iteration of* f*. Then for every* n *with* $m \prec n$*,* f *has some point of period* n*. In particular, if* f *has a point of period 3, then* f *has periods of all positive integer orders.*



Iterating the logistic map $f(x) = rx(1-x)$ from $x_0 = 0.4$, using $r = 3.7$:

$$0.4 \to 0.89 \to 0.36 \to 0.85 \to \ldots$$



Bifurcation diagrams showing long-term behaviour of the iterated logistic map for different r.

For any choice of parameter r, the *logistic map*, $f(x) = rx(1-x)$, certainly gives a continuous curve, as is illustrated in the left-hand plot (using $r = 3.7$). The long-term behaviour of the iterated logistic map is displayed in the right-hand plots: choose an arbitrary initial value $x_0 \in (0, 1)$ and calculate the sequence $f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots$; for values of r between zero and $r \approx 3.57$ the sequence leads our arbitrary initial value to convergence in a unique cycle through 2^i values, $i = 0, 1, 2, \ldots$ (period 2^i). Beyond 3.57, chaos ensues: indeed, the choice $r = 3.7$ appears to provide no convergent behaviour, and a tiny change to x_0 eventually causes a large change to our sequence. Suddenly, around $r = 3.83$, a window of order opens! On the far right, magnification shows convergence to a cycle of period three. But here Sharkovsky's theorem predicts periods of *any* length. Thus some choice of x_0 will converge to a cycle of length, say, 100; This cycle will not be an *attractor*: an infinitesimal change to x_0 will drive us back to period 3. Such non-attracting periods are invisible to computers!

The Ukranian O.M. Sharkovsky's 1964 theorem shows that another world exists behind the complexity of fractal plots.

Web link: www.mcasco.com/Order/ordintro.html: an excellent free on-line course by J.D. Jones for M. Casco Associates.

Further reading: *Over and Over Again* by Gengzhe Chang and Thomas W. Sederberg, Mathematical Association of America, 1998, chapter 23.