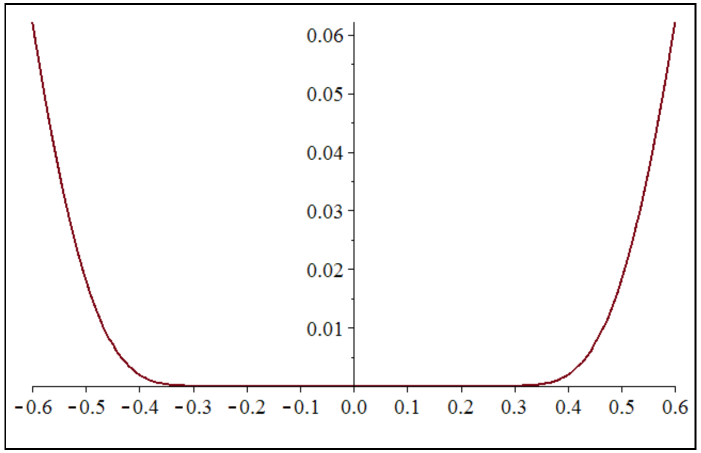
THEOREM OF THE DAY

Taylor's Theorem *Let c be a real number and f a real-valued function which is $(n+1)$-times differentiable in some interval I around c. Then for $x \in I$, there is some value θ lying between x and c, such that*

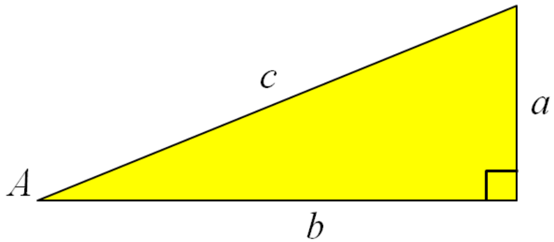
$$f(x) = f(c) + f'(c)(x-c) + f''(c)\frac{(x-c)^2}{2!} + \ldots + f^{(n)}(c)\frac{(x-c)^n}{n!} + f^{(n+1)}(\theta)\frac{(x-c)^{n+1}}{(n+1)!}.$$



The usual interpretation derives a 'power series expansion of $f(x)$ about $x = c$', perhaps most familiarly when $c = 0$ and we have the **Maclaurin Series**: $f(x) = \sum_{k=0}^{\infty} f^{(k)}(0)x^k/k!$. Whether the expansion converges depends on the behaviour, as $n \to \infty$, of the final 'remainder' term in the expression of the theorem. But there is much subtlety involved, as demonstrated by the famous example $f(x) = e^{-1/x^2}$, $f(0) = 0$, due to Cauchy and plotted left. It can be shown that every derivative of this function at $x = 0$ exists and is zero. So the Maclaurin series fails to distinguish $f(x)$ from the zero function! Nevertheless, Taylor's Theorem works perfectly well: if we take $n = 2$ and $c = 0$ we obtain

$$e^{-1/x^2} = 0 + 0 \times x + 0 \times \frac{x^2}{2!} + f^{(3)}(\theta)\frac{x^3}{3!} = \left(8\theta^{-9} - 36\theta^{-7} + 24\theta^{-5}\right)e^{-1/\theta^2}\frac{x^3}{6},$$

and we can solve for θ for a given x value, say $x = 1/2$ (for which $\theta \approx 0.2715442934149$, certainly lying between 0 and 1/2, gives e^{-4} to an accuracy of 13 decimal places).



Many functions, however, are well-defined by their Taylor or Maclaurin series which converge, if not everywhere ($e^x, \sin x, \cos x$) then at least close to the point of expansion ($\ln(1-x), (1-x)^{-1}, \sin^{-1} x$). An example is **Gregory's Series**: $\tan^{-1} x = x - x^3/3 + x^5/5 - x^7/7 + \ldots$, which converges for $|x| \leq 1$. We may find a different function whose expansion initially 'shadows' Gregory's series but eventually it will depart from it: an example is $f(x) = 3x\big/\left(1 + 2\sqrt{1+x^2}\right)$ which has Maclaurin series $x - x^3/3 + 7x^5/36 - \ldots$. Such shadowing may provide a neat rule of thumb: suppose we have a right triangle with sides $a \leq b < c$, as shown. Then, dividing each side by b and noting that $a/b \leq 1$, we have $A = \tan^{-1}(a/b) \approx 3(a/b)\big/\left(1 + 2\sqrt{1+(a/b)^2}\right)$, and arrive at what may be termed **Hugh Worthington's Rule:** $A \approx 3a/(b+2c)$ (measuring in radians).

Brook Taylor published his theorem in his *Methodus incrementorum directa et inversa* of 1715 and it was popularised by Colin Maclaurin in his 1742 *Treatise of Fluxions*, but the idea was known to James Gregory in the 1670s and to other pioneers of the calculus, while a rigorous understanding had to wait at least until Cauchy's work in the 1820s. The version given here, explicitly identifying a remainder term, is due to Lagrange in the early 1800s. Hugh Worthington's rule appears in "An essay on the resolution of plain triangles", 1780.

Web link: Lecture 14 at npflueger.people.amherst.edu/math1b/. See pballew.blogspot.co.uk/2014/07/a-curious-geometry-relation-and-question.html regarding Worthington's rule.

Further reading: *A Radical Approach to Real Analysis, 2nd edition* by David M. Bressoud, Mathematical Association of America, 2007, chapter 2. An extract of Worthington's essay is included in *A Wealth of Numbers: An Anthology of 500 Years of Popular Mathematics Writing* by Benjamin Wardhaugh, Princeton University Press, 2012, chapter 4.