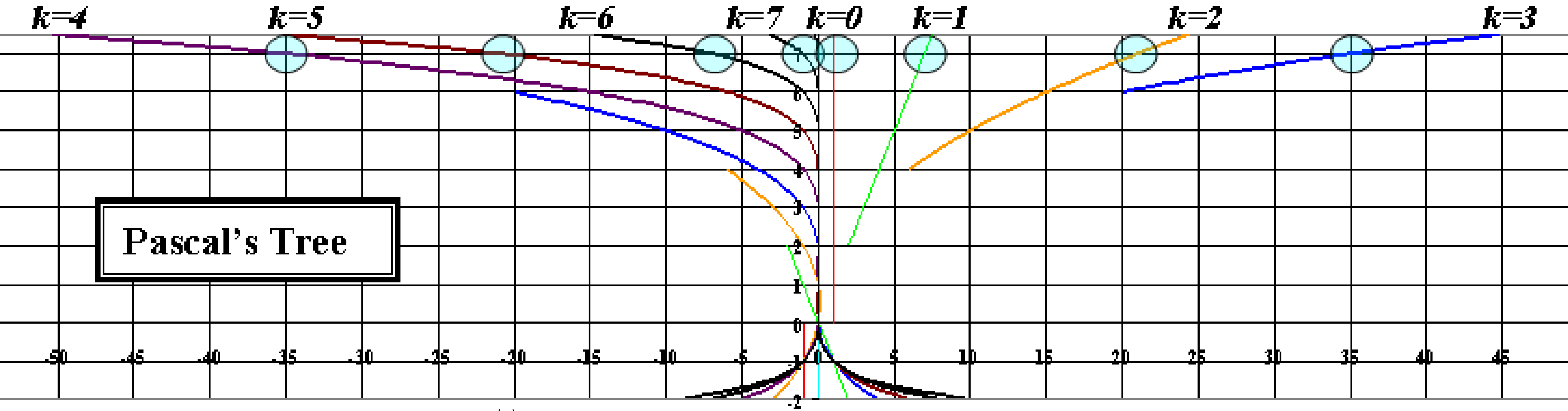
THEOREM OF THE DAY

The Binomial Theorem *For n a positive integer and real-valued variables x and y,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$



Given n distinct objects, the binomial coefficient $\binom{n}{k} = n!/k!(n-k)!$ counts the number of ways of choosing k. Transcending its combinatorial role, we may instead write the binomial coefficient as: $\binom{n}{k} = \frac{n}{k} \times \frac{n-1}{k-1} \times \cdots \times \frac{n-(k-1)}{1}$; taking $\binom{n}{0} = 1$. This form is defined when n is a real or even a complex number. This generalises the binomial theorem, with the summation counting over all nonnegative integers k.

In the above graph, n is a real number, and increases continuously on the vertical axis from -2 to 7.5. For different values of k, the value of $\binom{n}{k}$ has been plotted but with its sign reversed on reaching $n = 2k$, giving a discontinuity. This has the effect of spreading the binomial coefficients out on either side of the vertical axis: we recover, for integer n, a sort of (upside down) Pascal's Triangle. The values of the triangle for $n = 7$ have been circled.

When the right-hand summation in the theorem is extended to $k = \infty$, the theorem requires that the summation converges. This is guaranteed when n is an integer or when $|y/x| < 1$, so that, for instance, summing for $(4+1)^{1/2}$ gives a method of calculating $\sqrt{5}$.

The binomial theorem may have been known, as a calculation of poetic metre, to the Hindu scholar Pingala in the 5th century BC. It can certainly be dated to the 10th century AD. The extension to complex exponent n, using generalised binomial coefficients, is usually credited to Isaac Newton.

Web link: arxiv.org/abs/1105.3513

Further reading: *A Primer of Real Analytic Functions, 2nd ed.* by Steven G. Krantz and Harold R. Parks, Birkhäuser Verlag AG, 2002, section 1.5.