Theorem of the Day

The Binomial Theorem For \( n \) a positive integer and real-valued variables \( x \) and \( y \),

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.
\]

Given \( n \) distinct objects, the binomial coefficient \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \) counts the number of ways of choosing \( k \). Transcending its combinatorial role, we may instead write the binomial coefficient as:

\[
\binom{n}{k} = \frac{n}{k} \times \frac{n-1}{k-1} \times \cdots \times \frac{n-(k-1)}{1} ;
\]

taking \( \binom{n}{0} = 1 \). This form is defined when \( n \) is a real or even a complex number.

In the above graph, \( n \) is a real number, and increases continuously on the vertical axis from -2 to 7.5. For different values of \( k \), the value of \( \binom{n}{k} \) has been plotted but with its sign reversed on reaching \( n = 2k \), giving a discontinuity. This has the effect of spreading the binomial coefficients out on either side of the vertical axis: we recover, for integer \( n \), a sort of (upside down) Pascal’s Triangle. The values of the triangle for \( n = 7 \) have been circled.

If the right-hand summation in the theorem is extended to \( k = \infty \), the result still holds, provided the summation converges. This is guaranteed when \( n \) is an integer or when \(|y/x| < 1\), so that, for instance, summing for \((4 + 1)^{1/2}\) gives a method of calculating \( \sqrt{5} \).

The binomial theorem may have been known, as a calculation of poetic metre, to the Hindu scholar Pingala in the 5th century BC. It can certainly be dated to the 10th century AD. The extension to complex exponent \( n \), using generalised binomial coefficients, is usually credited to Isaac Newton.

Web link: arxiv.org/abs/1105.3513

Further reading: A Primer of Real Analytic Functions, 2nd ed. by Steven G. Krantz and Harold R. Parks, Birkhäuser Verlag AG, 2002, section 1.5.