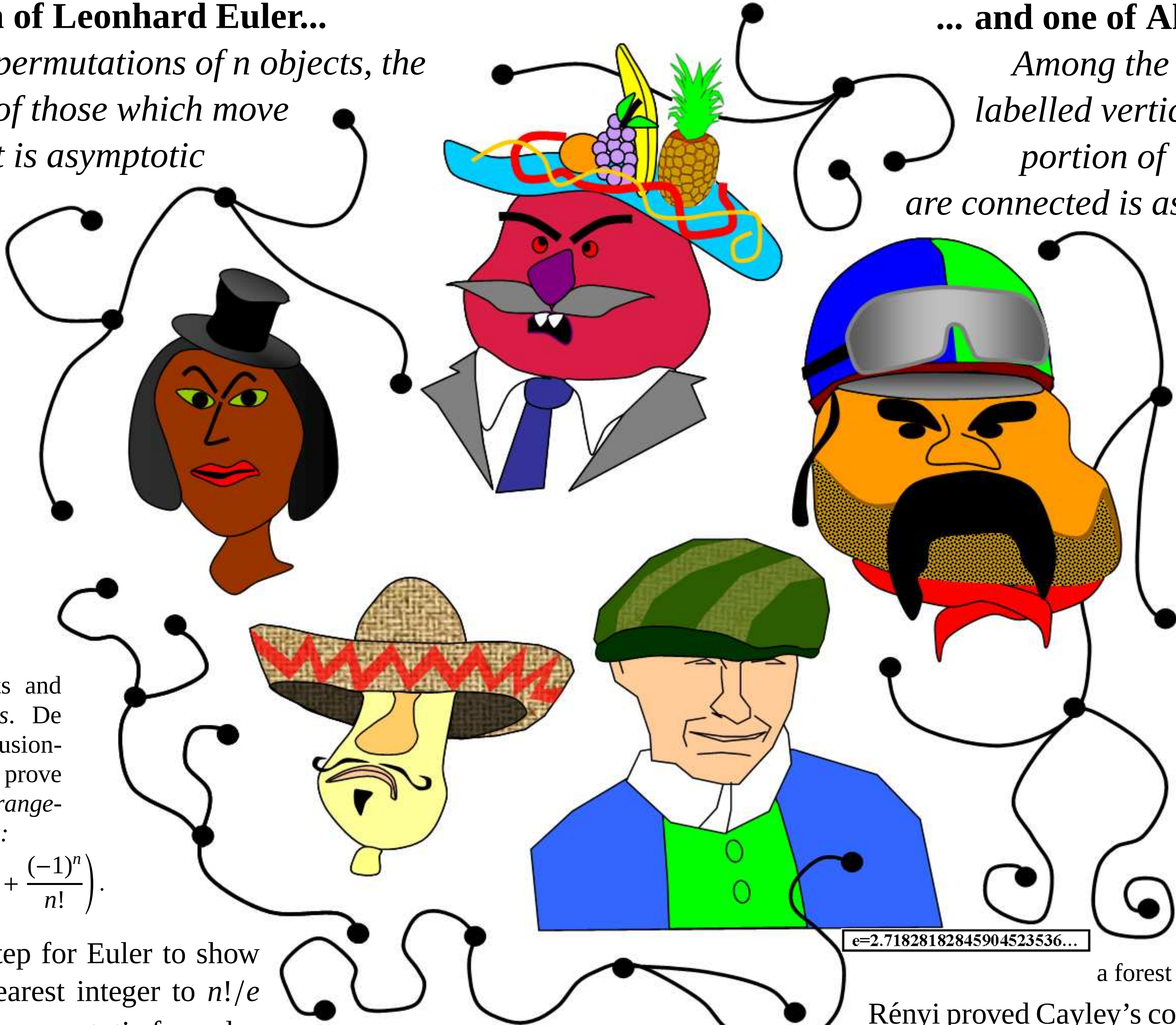
THEOREM OF THE DAY

A Theorem of Leonhard Euler...

Among the permutations of n objects, the proportion of those which move every object is asymptotic to $\dfrac{1}{e}$.

... and one of Alfréd Rényi

Among the forests on n labelled vertices, the proportion of those which are connected is asymptotic to $\dfrac{1}{\sqrt{e}}$.



In 1708, the French mathematician Pierre de Montmort asked how many ways the hats of his guests could be returned to them so that everybody got the wrong hat. These are permutations which move every one of n objects and are called *derangements*. De Montmort used the inclusion-exclusion principle to prove that *the number of derangements d_n for n objects is:*

$$d_n = n!\left(1 - \frac{1}{1!} + \ldots + \frac{(-1)^n}{n!}\right).$$

It was a short step for Euler to show that d_n is the nearest integer to $n!/e$ and to derive his asymptotic formula.

In 1889 Arthur Cayley considered a wonderful formula of Carl Wilhelm Borchardt: the number of trees (connected graphs or networks without cyclic paths) on n nodes labelled $1, \ldots, n$ is $t_n = n^{n-2}$. He proposed that the number of collections of k separate trees (*forests*) on $1, \ldots, n$, such that each of $1, \ldots, k$ appeared in a different tree would be

$$f_{n,k} = kn^{n-1-k}.$$

(The picture features a forest with $n = 29$ and $k = 5$.)

Rényi proved Cayley's conjecture in 1959 and showed that counting over all k gave an asymptotic total of $\sqrt{e}n^{n-2}$ forests.

Web link: www-math.mit.edu/~rstan/algcomb/ — appendix to chapter 9.

Further reading: *"e", The Story of a Number* by Eli Maor, Princeton University Press, 1998.