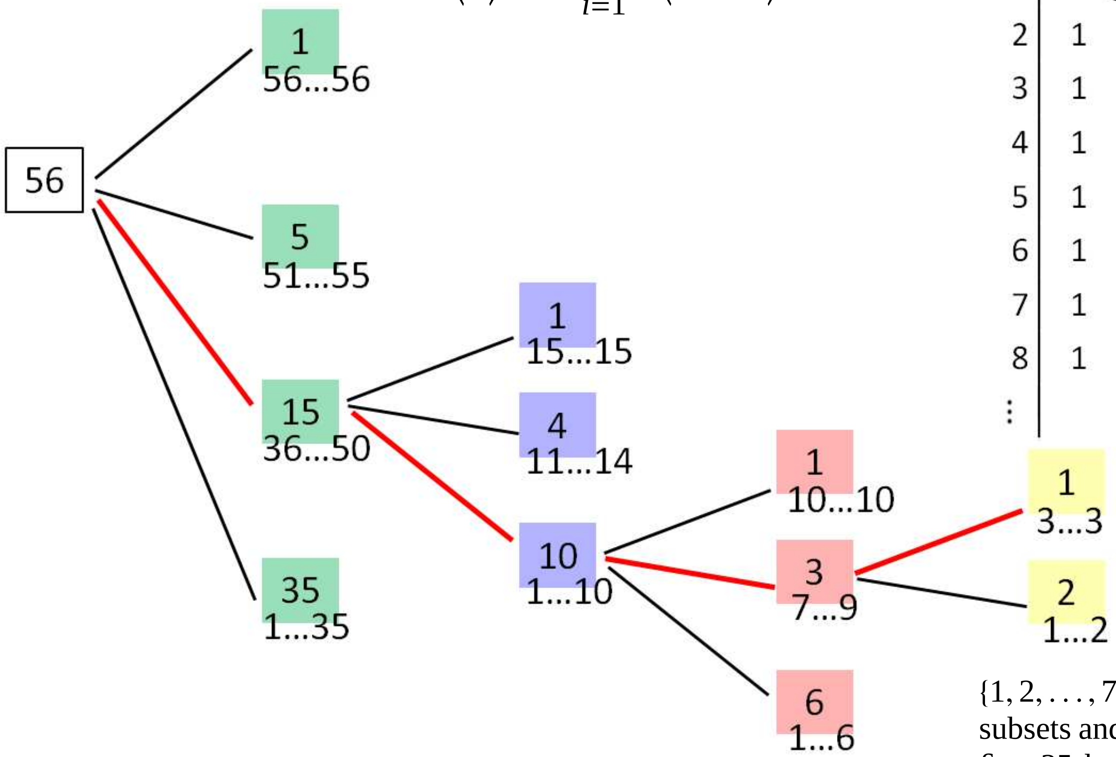
THEOREM OF THE DAY

The Hockey Stick Identity *The number of ways of choosing k elements from a set of n, with $n, k \geq 1$, may be calculated as:*

$$\binom{n}{k} = \sum_{i=1}^{n-k+1} \binom{n-i}{k-1}.$$



	0	1	2	3	4	5	6	7	8	...
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	70	56	28	8	1	
⋮										

In the illustration of Pascal's triangle to the left, we may start at $\binom{8}{5} = 56$ and count back up the 'hockey stick' to confirm that $56 = 35 + 15 + 5 + 1$. **Proof of the identity:** list the n elements of the set and suppose that the i-th is the first of the k chosen elements. This leaves $n - i$ elements from which $k - 1$ remain to be chosen, that is $\binom{n-i}{k-1}$ choices. We start at $i = 1$ and continue until $i = n - (k - 1)$. That is the last value of i which leaves at least $k - 1$ available elements. So the total number of choices is $\sum_{i=1}^{n-k+1} \binom{n-i}{k-1}$.

The hockey stick identity, despite its simplicity, is a versatile tool in combinatorics and probability. Here we illustrate its use in **simple random sampling**: choosing k elements from n in such a way that every k-element subset is chosen with equal probability. Thus, suppose our population is the set $\{1, 2, \ldots, 7, 8\}$ and from this we want to choose a subset of 5. There are $\binom{8}{5} = 56$ possible subsets and these may be listed in order as they appear in the above hockey stick proof: the first 35 have smallest element 1; the next 15 have smallest element 2; the next 5 have smallest element 3; and the final 1 has smallest element 4. This is depicted above left as a 4-way branching from a root node labelled 56. Generate a random number between 1 and 56: it will select a branch in proportion to the number of subsets it represents. For example, suppose our random number is 42. This indicates one of the 15 subsets whose smallest element is 2. Take this branch. Now we repeat the selection process using the hockey stick which goes up from $\binom{6}{4} = 15$ in Pascal's triangle. And we continue recursively until our hockey sticks reach the left-hand edge of the triangle. You may check that, in our illustration, we have chosen the subset $\{2, 3, 5, 7, 8\}$.

The hockey stick is explicit in the famous 1665 *Traité* of Pascal but also in the 1570 *Opus novum de proportionibus* of Cardano, and a convincing argument can be made for it being known to Chu Shih-Chieh around 1300.

Web link: thousandmaths.tumblr.com/post/127167254744/chus-theorem-i
Further reading: Counting, 2nd edition by Khee Meng Koh and Eng Guan Tay, World Scientific, chapter 12.