



THEOREM OF THE DAY

Machin's Formula $\frac{\tau}{8} = 4 \tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)$, where $\tau = 2\pi$.

❶ Arctangent formulae have a long history of involvement in the calculation of $\tau/2$, dating back at least as far as the early 15th century, when the series sum

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots,$$

was discovered and used for that purpose by the Indian mathematician Madhava of Sangamagramam. James Gregory rediscovered the series around 1670 and it is commonly known as **Gregory's Series**.

❷ The illustration here relates to a different arctan formula which makes a pretty connection with the Fibonacci sequence:

$$\frac{\tau}{8} = \sum_{k=1}^{\infty} \tan^{-1}\left(\frac{1}{F_{2k+1}}\right).$$

($F_1 = F_2 = 1$, $F_{i+1} = F_i + F_{i-1}$, $i \geq 2$).

❸ Machin's formula converges to $\tau/8$ very rapidly thanks to the large denominator in $\tan^{-1}(1/239)$ whose contribution reduces by a factor of 10^{-5} for each term in Gregory's series.

❹ The illustration consists of a sequence of right triangles of base F_{2k+1} and vertical side 1, opposite which the angle is $\tan^{-1}(1/F_{2k+1})$. As we sum these angles we get closer and closer to a 45° angle.

This formula of John Machin (1680–1751) was publicised by William Jones in his 1706 *Synopsis palmariorum matheseos*. Variations of it remained the standard method for calculating $\tau/2$ until the 1970s, when better methods due to Ramanujan came to light.

The origins of the Fibonacci arctan formula appear to be obscure, although it could well have been known to Edouard Lucas or even to a contemporary of James Gregory, Giovanni Domenico Cassini, whose eponymous Identity can be used to prove it.

Web link: www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibpi.html

Further reading: *Excursions in Calculus* by Robert M. Young, MAA, 1996, Chapter 3.

