Pascal’s Triangle

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LSBU Maths Study Group
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1
1, 1
1, 2, 1
1, 3, 3, 1
1, 4, 6, 4, 1
1, 5, 10, 10, 5, 1
1, 6, 15, 20, 15, 6, 1
1, 7, 21, 35, 35, 21, 7, 1
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1, 18, 153, 816, 3060, 8568, 18564, 31824, 43758, 48620, 43758, 31824, 18564, 8568, 3060, 816, 153, 18, 1
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1, 22, 231, 1540, 7315, 26334, 74613, 170544, 319770, 497420, 646646, 705432, 646646, 497420, 319770, 170544, 74613, 26334, 7315, 1540, 231, 22, 1
1, 23, 253, 1771, 8855, 33649, 100947, 245157, 490314, 817190, 1144066, 1352078, 1352078, 1144066, 817190, 490314, 245157, 100947, 33649, 8855, 1771, 253, 23, 1
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1, 26, 325, 2600, 14950, 65780, 230230, 657800, 1562275, 3124550, 5311735, 7726160, 9657700, 10400600, 9657700, 7726160, 5311735, 3124550, 1562275, 657800, 230230, 65780, 14950, 2600, 325, 26, 1
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Binomial coefficients $\binom{n}{m}$

Defined by $\binom{n}{m}$ = number of ways of choosing m objects from n. Same as number of $m-$subsets of an n-set.
So sum of row n entries is 2^n.

Also: defined for $1 \le m < n$, by $\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}$, and by $\binom{n}{n} = \binom{n}{0} = 1$.

Also: defined by $\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{n\times(n-1)\times\cdots\times(n-m+1)}{m\times(m-1)\times\cdots\times 2\times 1}$
which remains defined for any real or even complex n, consistent with the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^{n-k} y^k,$$

$$\binom{n}{k} = 0, k > n.$$

1										
1	1									
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	210	252	210	120	45	10	1

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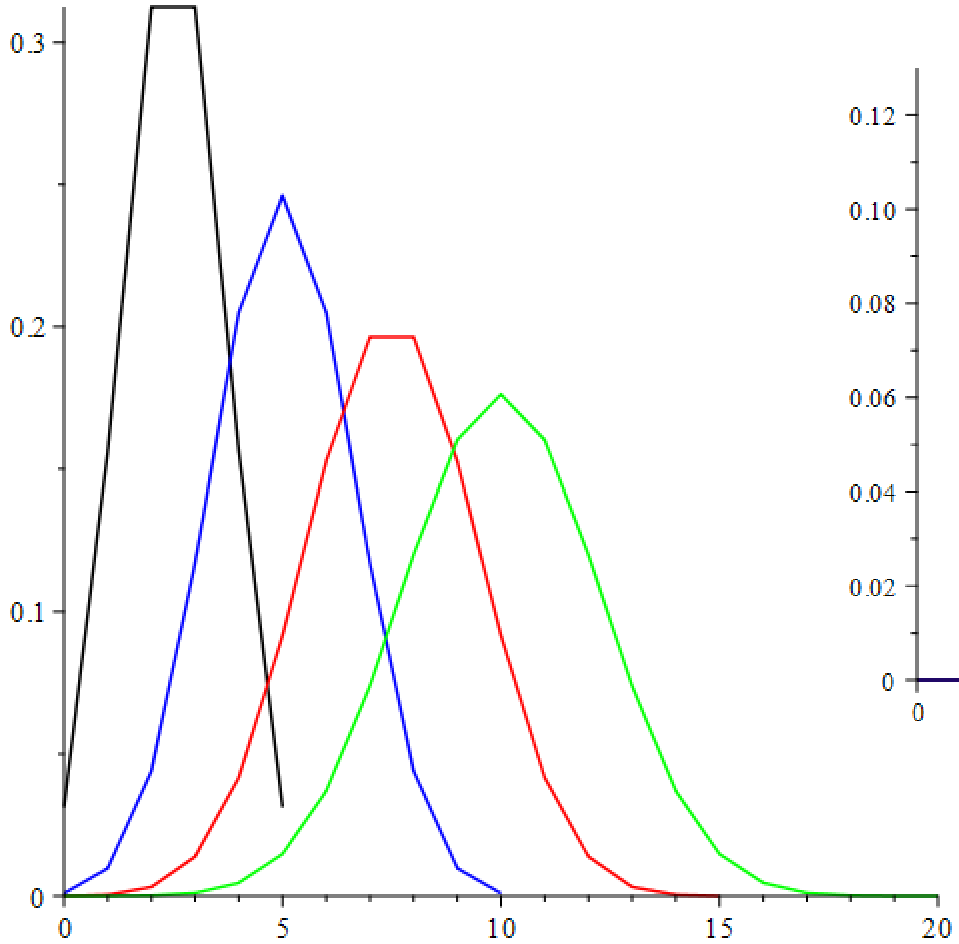
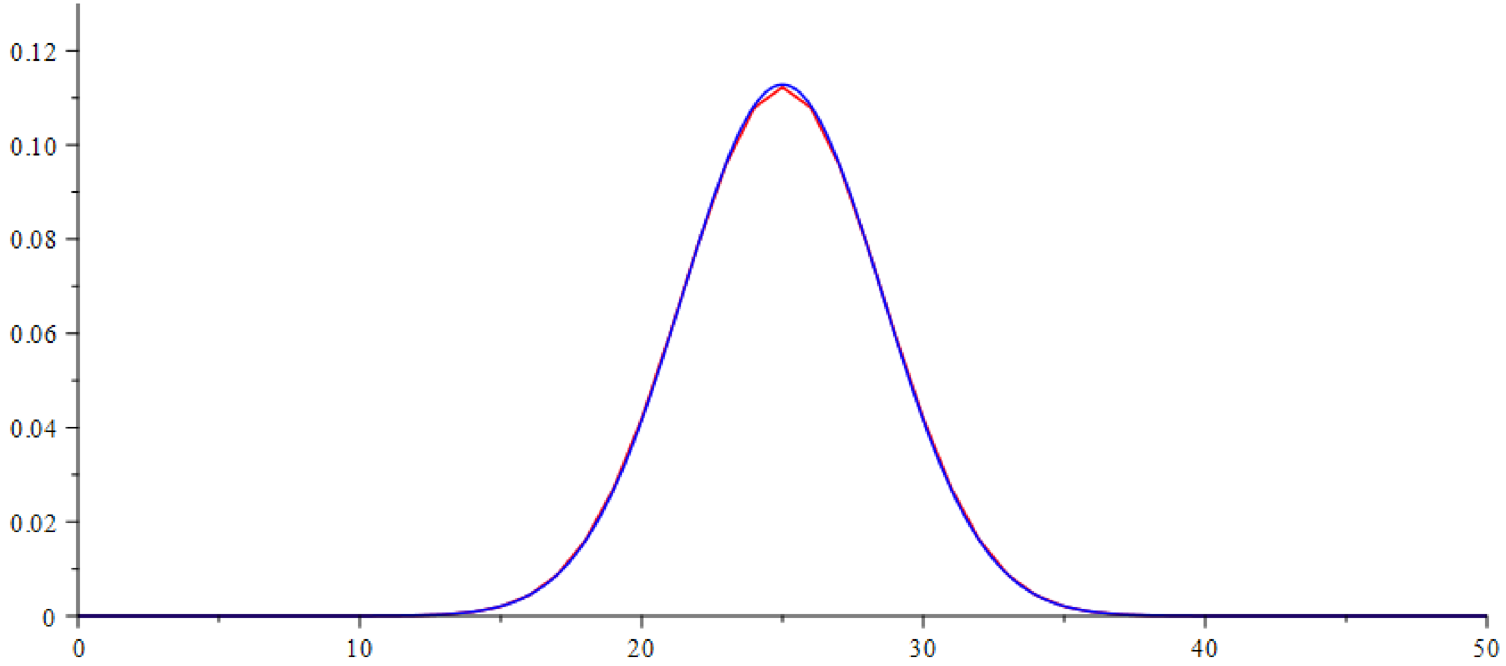
Normal curve approximation to binomial

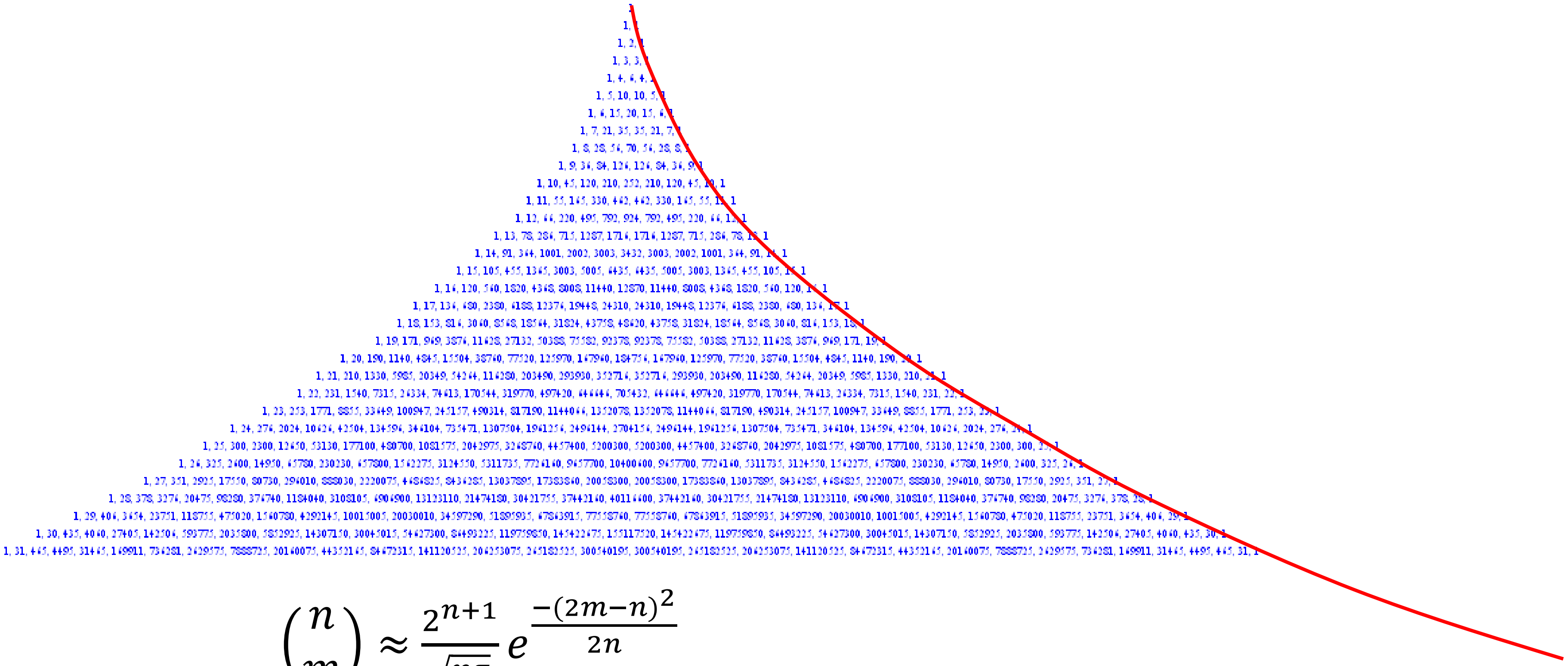
Plots of $\binom{n}{m}/2^n$ for n= 5,10,15,20





$\binom{n}{m}/2^n$ for n= 50, plotted against $\frac{2}{\sqrt{n\tau}}e^{\frac{-(2x-n)^2}{2n}}$
(blue curve)

What curve is this?



$$\binom{n}{m} \approx \frac{2^{n+1}}{\sqrt{n\tau}} e^{\frac{-(2m-n)^2}{2n}}$$

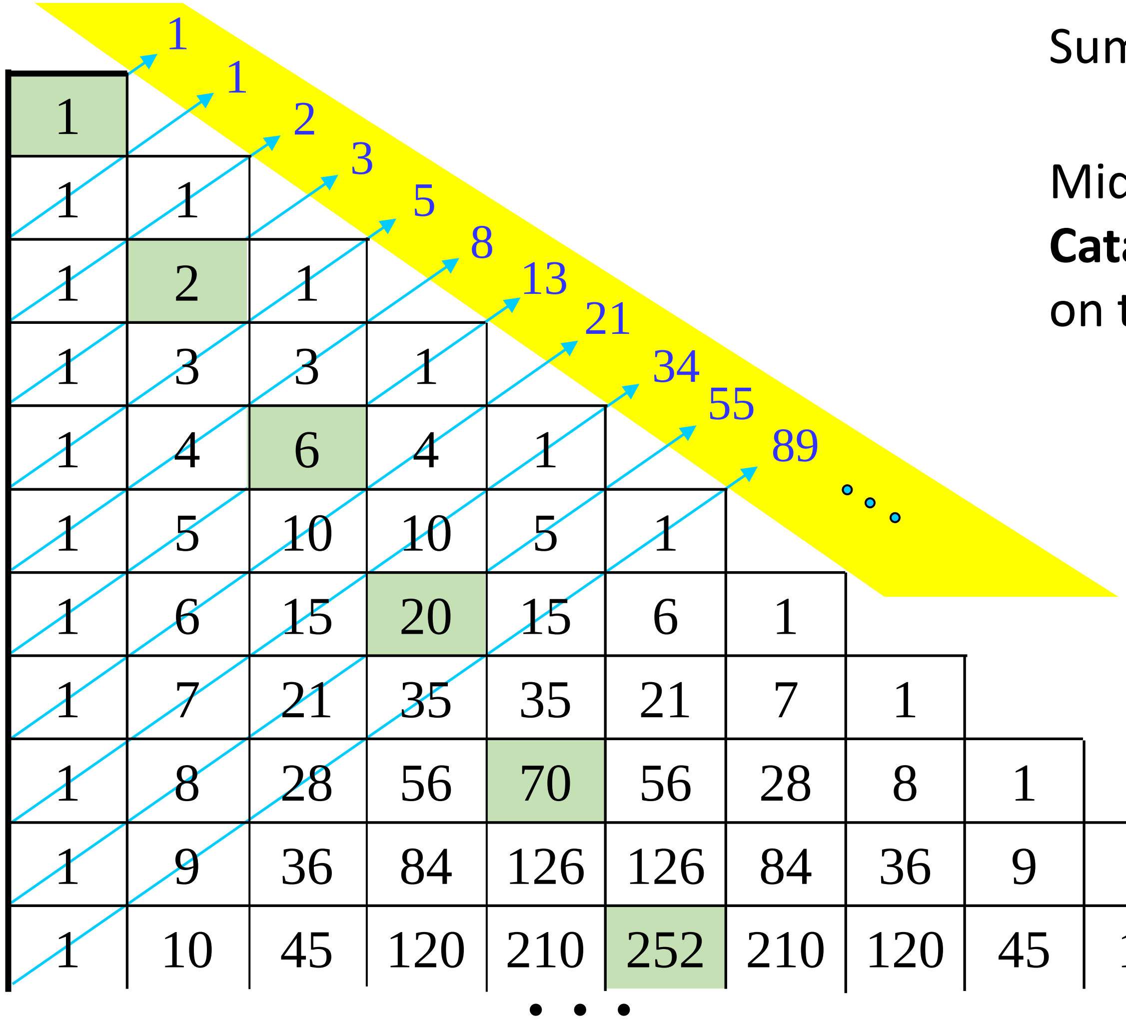
Take log of right-hand-side (to get number of digits)

Sum over m

Get highest term n^2

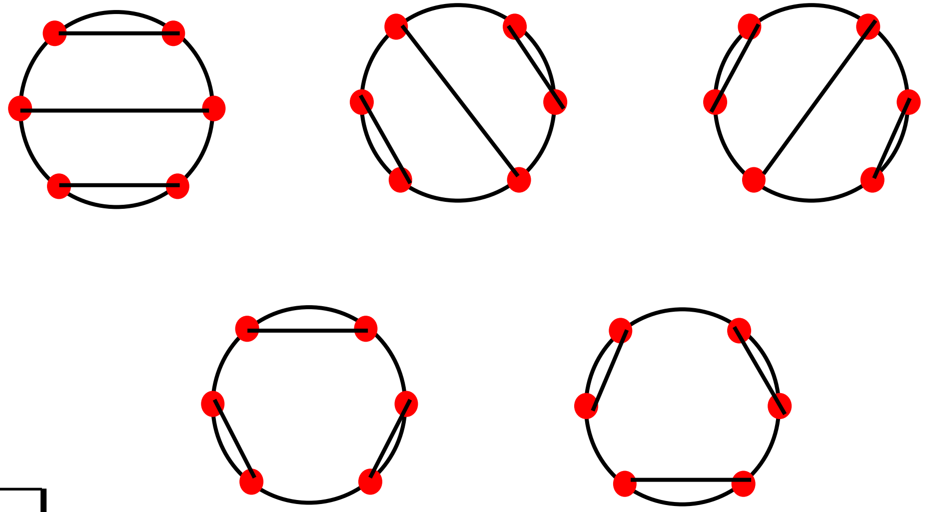
So this should be called Pascal's parabola!

Relationships to other combinatorial sequences



Sums of ‘diagonals’ are **Fibonacci numbers**

Middle element of row $2n$, divided by $n + 1$ is n-th **Catalan number**. E.g. number of ways to join $2n$ points on the circumference of a circle with n chords



Harlan J. Brothers' formula

A relationship between Pascal and Euler's number e appears to have first been discovered by Harlan J. Brothers in 2012.Let s_n = product of entries in row n of triangle. Then

$$\lim_{n\to\infty}\frac{s_{n-1}s_{n+1}}{s_n^2}=e.$$

1	**1**										
1	1	**1**									
1	2	1	**2**								
1	3	3	1	**9**							
1	4	6	4	1	**96**						
1	5	10	10	5	1	**2500**					
1	6	15	20	15	6	1	**162000**				
1	7	21	35	35	21	7	1	**26471025**			
1	8	28	56	70	56	28	8	1	**11014635520**		
1	9	36	84	126	126	84	36	9	1	**11759522374656**	
1	10	45	120	210	252	210	120	45	10	1	**3240609120000000**

• • •

$$\frac{S_7\times S_9}{S_8^2}=\frac{26471025\times 11759522374656}{11014635520^{\,2}}=\frac{311286610767578342400}{121322195638445670400}\approx 2.5658$$

$$\frac{S_{99}\times S_{101}}{S_{100}^2}\approx 2.7048.$$

Proof: a calculation shows that $\frac{s_{n-1}s_{n+1}}{s_n^2}=\left(1+\frac{1}{n}\right)^n$ which in the limit is equal to e.

Pascal and prime numbers I

Lucas's Theorem

If p is a prime number and if $m \le n$ are written in base p thus:

$m = \sum_{i=0}^{s} m_i p^i$ and $n = \sum_{j=0}^{t} n_j p^j, s \le t$, then

$$\binom{n}{m} \equiv \binom{n_0}{m_0}\binom{n_1}{m_1}\binom{n_2}{m_2}\cdots\binom{n_t}{m_t} \pmod{p}.$$

1															
1	1														
1	2	1													
1	3	3	1												
1	4	6	4	1											
1	5	10	10	5	1										
1	6	15	20	15	6	1									
1	7	21	35	35	21	7	1								
1	8	28	56	70	56	28	8	1							
1	9	36	84	126	126	84	36	9	1						
1	10	45	120	210	252	210	120	45	10	1					
1	11	55	165	330	462	462	330	165	55	11	1				
1	12	66	220	495	792	924	792	495	220	66	12	1			
1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1		
1	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	14	1	
1	15	105	455	1365	3003	5005	6435	6435	5005	3003	1365	455	105	15	
1	16	120	560	1820	4368	8008	11440	12870	11440	8008	4368	1820	560	120	
1	17	136	680	2380	6188	12376	19448	24310	24310	19448	12376	6188	2380	680	1
1	18	153	816	3060	8568	18564	31824	43758	48620	43758	31824	18564	8568	3060	8
1	19	171	969	3876	11628	27132	50388	75582	92378	92378	75582	50388	27132	11628	3
1	20	190	1140	4845	15504	38760	77520	125970	167960	184756	167960	125970	77520	38760	15
1	21	210	1330	5985	20349	54264	116280	203490	293930	352716	352716	293930	203490	116280	54
1	22	231	1540	7315	26334	74613	170544	319770	497420	646646	705432	646646	497420	319770	17
1	23	253	1771	8855	33649	100947	245157	490314	817190	1144066	1352078	1352078	1144066	817190	49
1	24	276	2024	10626	42504	134596	346104	735471	1307504	1961256	2496144	2704156	2496144	1961256	1307504
1	25	300	2300	12650	53130	177100	480700	1081575	2042975	3268760	4457400	5200300	5200300	4457400	3268760
1	26	325	2600	14950	65780	230230	657800	1562275	3124550	5311735	7726160	9657700	10400600	9657700	7726160

E.g.: $\binom{24}{11} = 2496144 \equiv 4 \pmod{5}$.

$24 = 4 \times 5^0 + 4 \times 5^1$

$11 = 1 \times 5^0 + 2 \times 5^1$

and

$$\binom{4}{1} \times \binom{4}{2} = 24 \equiv 4 \pmod{5}.$$

Corollary: for prime p we have,

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p}.$$

Indeed, $\binom{n}{m} \equiv \binom{pn}{pm} \pmod{p}$, since multiplying by p appends a zero digit to the base p representation of any number.

Now set $n = 2, m = 1$ for

$$2 = \binom{2}{1} \equiv \binom{2p}{p} = 2\binom{2p-1}{p-1} \pmod{p}.$$

Pascal and prime numbers II

Wolstenholme's Theorem If $p \geq 5$ is prime, then

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}.$$

More generally, for prime $p \geq 5$, and positive integers n, m,

$$\binom{n}{m} \equiv \binom{pn}{pm} \pmod{p^3}$$

E.g., $4 = \binom{4}{3} \equiv \binom{5 \times 4}{5 \times 3} = 15504 \pmod{125}$ (the blue boxed entries).

However, this is not in general true for mod p^4, the two smallest (and only known) so-called Wolstenholme primes being 16843 and 2124679.

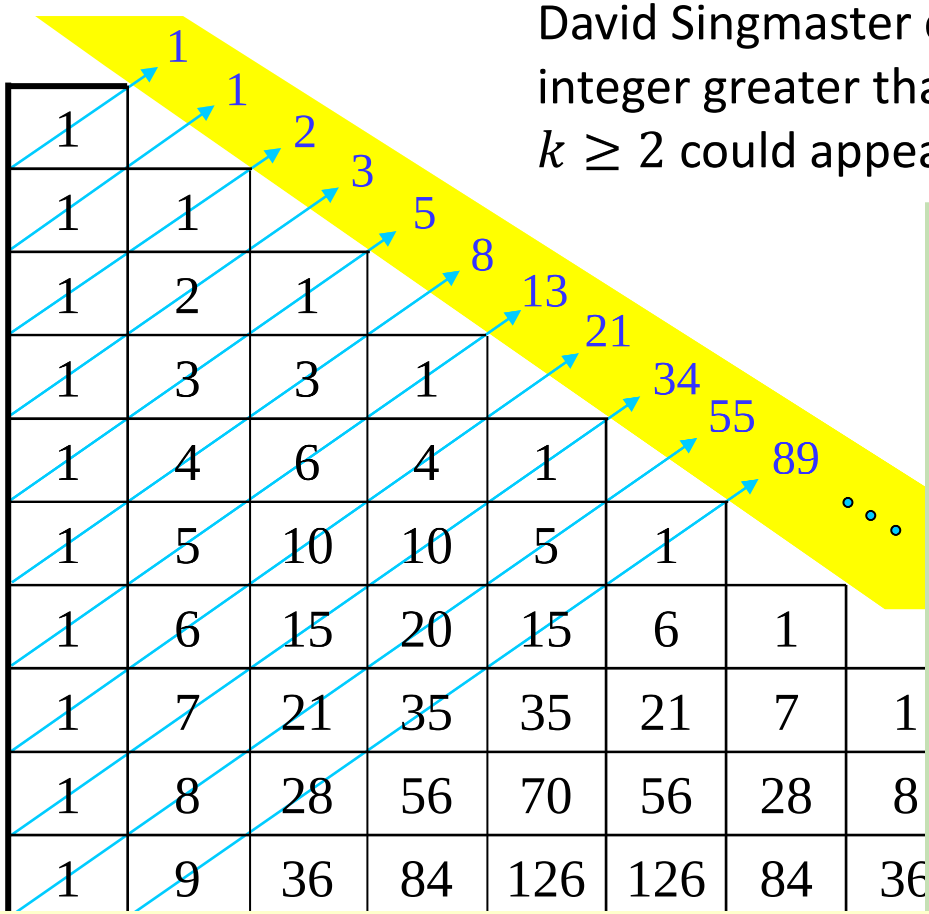
Open question: does there exist a composite n for which

$$\binom{2n-1}{n-1} \equiv 1 \pmod{n^3}?$$

If so $n \geq 10^9$.

1															
1	1														
1	2	1													
1	3	3	1												
1	4	6	4	1											
1	5	10	10	5	1										
1	6	15	20	15	6	1									
1	7	21	35	35	21	7	1								
1	8	28	56	70	56	28	8	1							
1	9	36	84	126	126	84	36	9	1						
1	10	45	120	210	252	210	120	45	10	1					
1	11	55	165	330	462	462	330	165	55	11					
1	12	66	220	495	792	924	792	495	220	66					
1	13	78	286	715	1287	1716	1716	1287	715	286					
1	14	91	364	1001	2002	3003	3432	3003	2002	1001					
1	15	105	455	1365	3003	5005	6435	6435	5005	3003	1365	455	105	15	1
1	16	120	560	1820	4368	8008	11440	12870	11440	8008	4368	1820	560	120	16
1	17	136	680	2380	6188	12376	19448	24310	24310	19448	12376	6188	2380	680	136
1	18	153	816	3060	8568	18564	31824	43758	48620	43758	31824	18564	8568	3060	816
1	19	171	969	3876	11628	27132	50388	75582	92378	92378	75582	50388	27132	11628	3876
1	20	190	1140	4845	15504	38760	77520	125970	167960	184756	167960	125970	77520	38760	15504
1	21	210	1330	5985	20349	54264	116280	203490	293930	352716	352716	293930	203490	116280	54264
1	22	231	1540	7315	26334	74613	170544	319770	497420	646646	705432	646646	497420	319770	170544
1	23	253	1771	8855	33649	100947	245157	490314	817190	1144066	1352078	1352078	1144066	817190	490314
1	24	276	2024	10626	42504	134596	346104	735471	1307504	1961256	2496144	2704156	2496144	1961256	1307504
1	25	300	2300	12650	53130	177100	480700	1081575	2042975	3268760	4457400	5200300	5200300	4457400	3268760
1	26	325	2600	14950	65780	230230	657800	1562275	3124550	5311735	7726160	9657700	10400600	9657700	7726160

Singmaster's conjecture



David Singmaster conjectured in 1971 that there was an absolute constant N such that no integer greater than 1 could appear in Pascal's triangle more than N times. He proved that $k \geq 2$ could appear at most $O(\log k)$ times and showed that $N = 8$ was valid up to $\mathrm{k} = 2^{48}$.

An infinite class of entries which occur 6 times is given by the identity

$$\binom{F_n F_{n+1} - 1}{F_{n-1} F_n} = \binom{F_n F_{n+1}}{F_{n-1} F_n - 1}.$$

E.g. $n \geq 4$ gives

$$\binom{F_4 F_5 - 1}{F_3 F_4} = \binom{14}{6} = \binom{F_4 F_5}{F_3 F_4 - 1} = \binom{15}{5} = 3003.$$

Since $\binom{14}{6} = \binom{14}{8}$ and $\binom{15}{5} = \binom{15}{10}$, and 3003 appears twice in row 3003 this gives 6 occurrences.

In fact it happens that 3003 occurs twice more, being the value of $\binom{78}{2} = \binom{78}{76}$.

The integer 3003 seems to be the only one occurring more than 6 times.

In 2021, Kaisa Matomäki, Maksym Radziwill, Xuancheng Shao, Terence Tao and Joni Teräväinen announced a proof of Singmaster's conjecture for the 'interior' of Pascal's triangle.

More precisely they showed that the number of solutions to the equation $\binom{n}{m} = t, t \geq 2$, is bounded by an absolute constant provided this is true in the region

$$2 \leq m < \exp\left(\log^{2/3+\epsilon} n\right).$$

A014430: Pascal −1

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Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A014430 Subtract 1 from Pascal's triangle, read by rows. 9

1, 2, 2, 3, 5, 3, 4, 9, 9, 4, 5, 14, 19, 14, 5, 6, 20, 34, 34, 20, 6, 7, 27, 55, 69, 55, 27, 7, 8, 35, 83, 125, 125, 83, 35, 8, 9, 44, 119, 209, 251, 209, 119, 44, 9, 10, 54, 164, 329, 461, 461, 329, 164, 54, 10, 11, 65, 219, 494, 791, 923, 791, 494, 219, 65, 11 (list; table; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS Each value of the sequence (T(x,y)) is equal to the sum of all values in Pascal's Triangle that are in the rectangle defined by the tip (0,0) and the position (x,y). - Florian Kleedorfer (florian.kleedorfer(AT)austria.fm), May 23 2005

To clarify A014430 and A129696: We subtract I = Identity matrix from Pascal's triangle to obtain the beheaded variant, A074909. Then take column sums starting from the top of A074909 to get triangle A014430. Row sums of the inverse of triangle A014430 gives the Bernoulli numbers, A027641/A026642. Alternatively, triangle A014430 as an infinite lower triangular matrix * [the Bernoulli numbers as a vector] = [1, 1, 1, ...]. Given the B_n version starting (1, 1/2, 1/6, ...) triangle A014430 * the B_n vector [1, 1/2, 1/6, 0, -1/30, ...] = the triangular numbers. - Gary W. Adamson, Mar 13 2012

If regarded as a symmetric array of the form

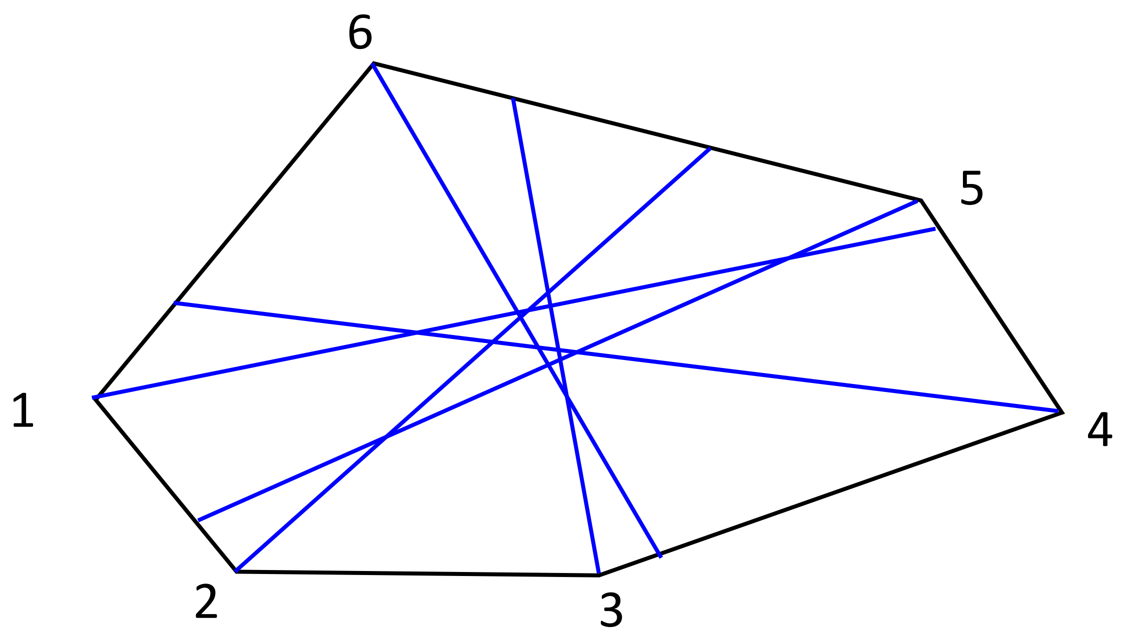
```
1  2   3   4    5  ...
2  5   9  14   20  ...
3  9  19  34   55  ...
4 14  34  69  125  ...
5 20  55 125  251  ...
6 27  83 209  461  ...
7 35 119 329  791  ...
8 44 164 494 1286  ...
9 54 219 714 2001  ...
```

it contains the rows (and columns) A000096, A062748, A063258, A062988, A124089, ..., A035927 and so on and counts the multisets of digits of numbers in base b>=2 with d>=1 digits (equivalent to the comment in A035927). - R. J. Mathar, Apr 25 2016

Proof of Florian Kleedorfer's formula: Take sums of the columns of the rectangle -

	1	2	3	4	5	6	...
0	0						
1	0	0					
2	0	1	0				
3	0	2	2	0			
4	0	3	5	3	0		
5	0	4	9	9	4	0	
6	0	5	14	19	14	5	0
⋮							

Bisecting convex polygons

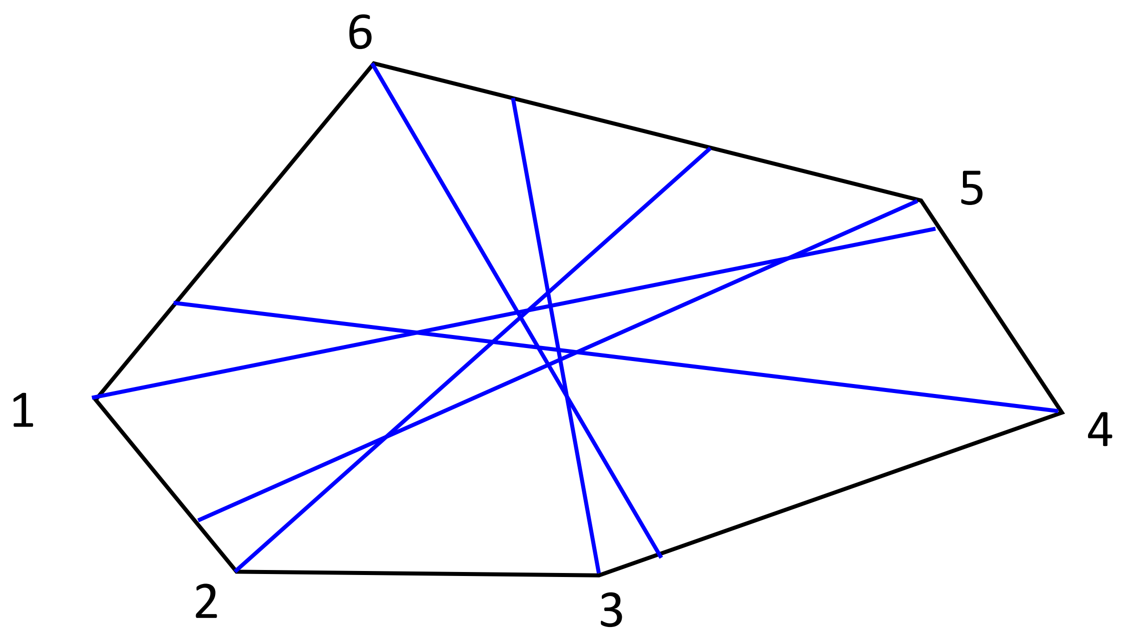


We find the straight line from each vertex to an opposite side which bisects the polygon area.

We take the vertices in anti-clockwise order and denote a bisecting edge from vertex *u* to edge *vw* by the ordered pair (*u*,*v*). So our example is

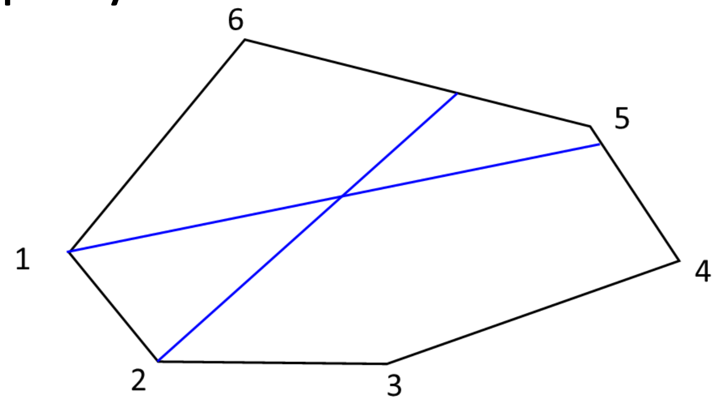
1, 4
2, 5
3, 5
4, 6
5, 1
6, 3

Pairs of bisecting lines



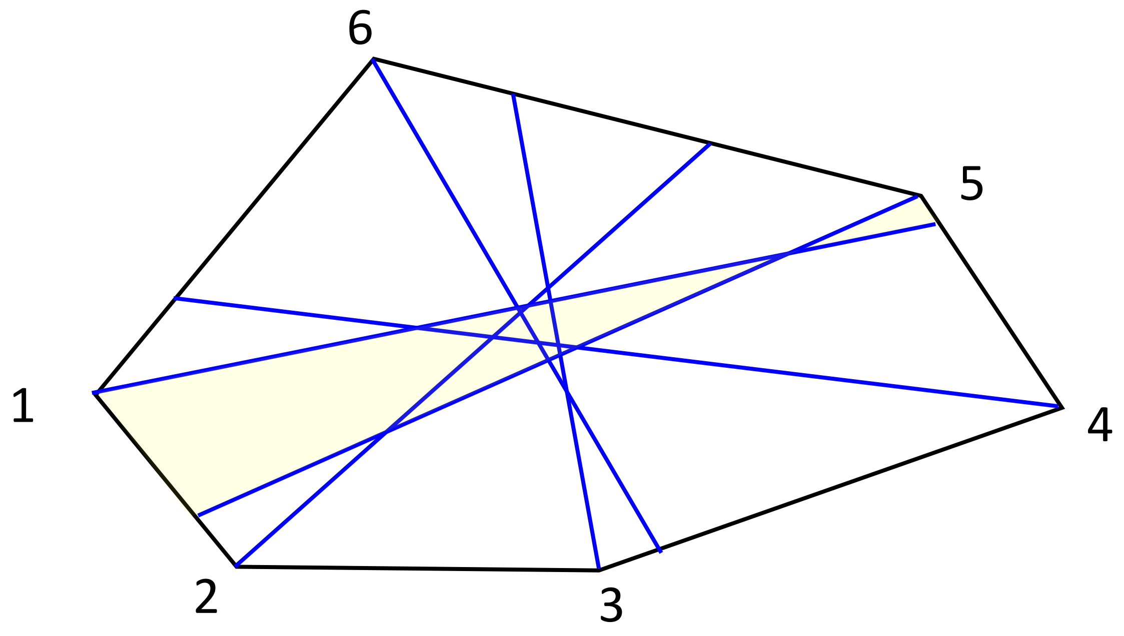
1, 4
2, 5
3, 5
4, 6
5, 1
6, 3

We want to take consecutive pairs of bisecting lines, forming a sector of the circle.
We want to avoid sectors which properly contain a vertex.



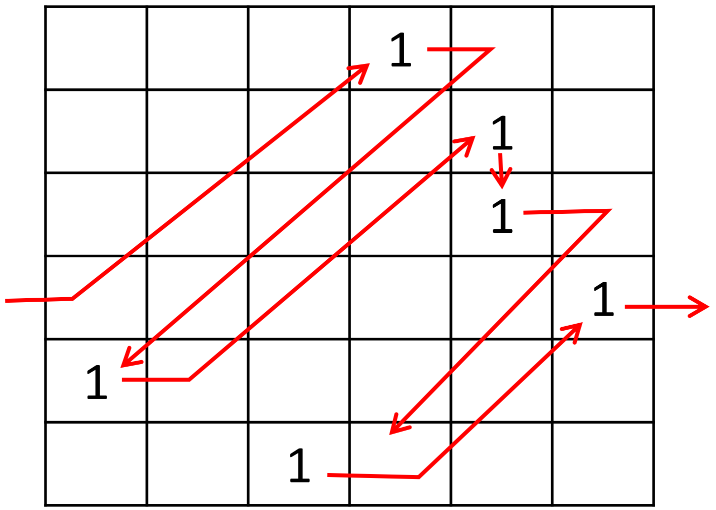
So we want to take the lines in an ordering which avoids this

Ordering bisecting line pairs



1, 4	1, 4
2, 5	5, 1
3, 5	2, 5
4, 6	3, 5
5, 1	6, 3
6, 3	4, 6

Represent the lines as matrix entries. The matrix lives on a torus, so it wraps round horizontally and vertically.

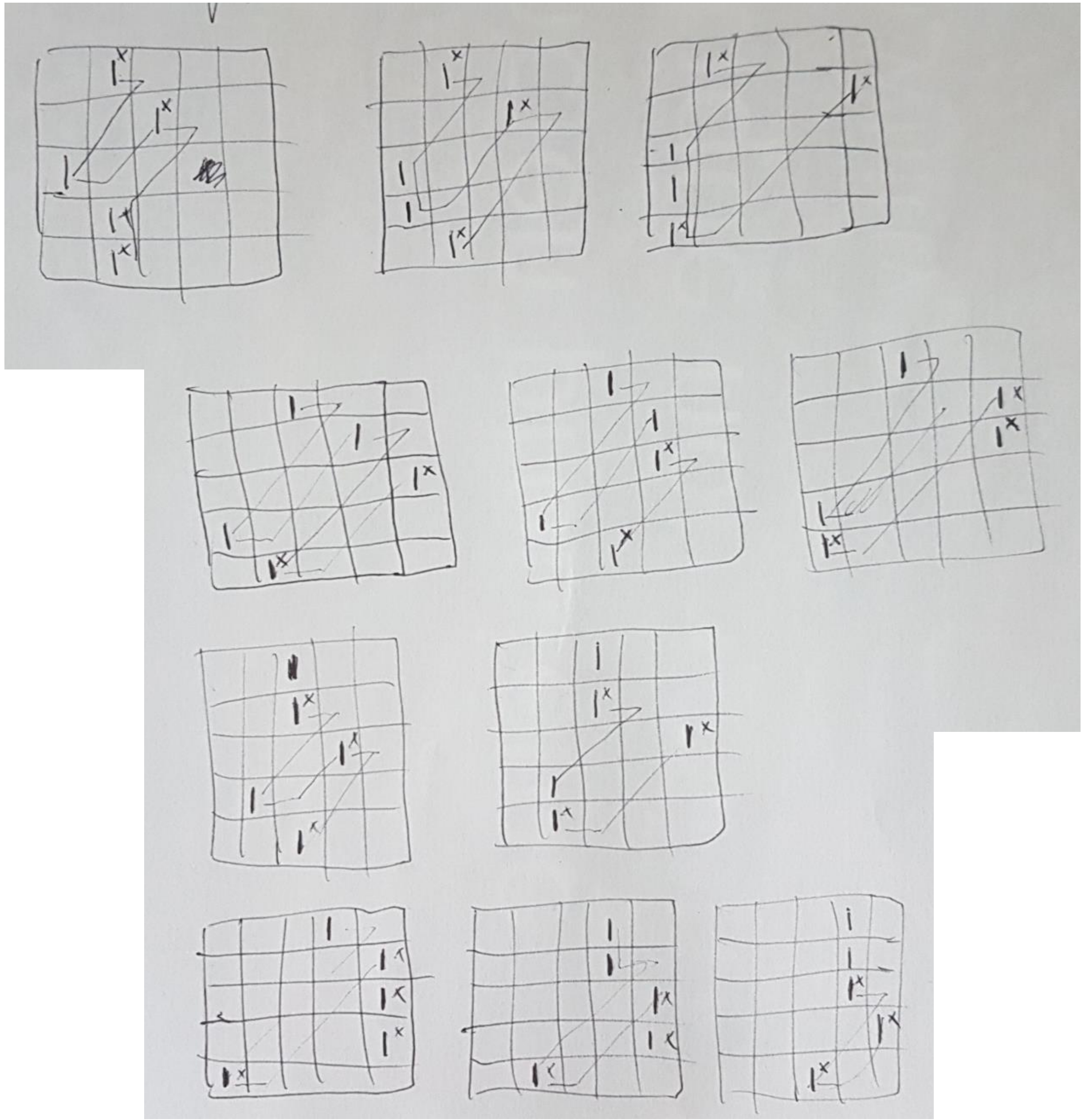


Connect the entries in a cycle using the following rules

1. a column of 1s is joined vertically
2. a 1 with a vacant cell below is joined to the entry diagonally opposite the cell to its right.

The result is the ordering we want:

Enumerating cyclic bisecting line matrices



3 starting in cell 2 of row 1

5 starting in cell 3 of row 1

3 starting in cell 4 of row 1