THEOREM OF THE DAY

A Theorem on Modular Fibonacci Periodicity Suppose that \((F_n)_{n \geq 0}\) is the sequence of Fibonacci numbers defined by \(F_0 = 0\), \(F_1 = 1\) and, for \(n \geq 2\), \(F_n = F_{n-1} + F_{n-2}\). Let \(m\) be a positive integer and let \((f_n)_{n \geq 0}\) be the Fibonacci sequence taken modulo \(m\), i.e. for \(n \geq 0\), let \(f_n = F_n \pmod{m}\). Then

1. the sequence \((f_n)\) takes the value zero periodically, with period, say, \(d(m)\);
2. the sequence \((f_n)\) itself is periodic with period, say, \(\pi(m)\). Moreover \(\pi(m)\) is even when \(m > 2\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F_n)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
<td>55</td>
<td>89</td>
<td>144</td>
<td>233</td>
<td>377</td>
<td>610</td>
<td>987</td>
<td>1597</td>
<td>2584</td>
<td>4181</td>
<td>6765</td>
<td>10946</td>
</tr>
<tr>
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<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
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<td>0</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>mod 4</td>
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<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
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<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
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<td>2</td>
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<td>3</td>
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<td>2</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

The values of \(F_0, \ldots, F_{21}\) are tabulated above, with reduction by various moduli. We observe that, in the notation of the theorem, \(d(2) = \pi(2) = 3\) and \(d(4) = \pi(4) = 6\); for \(m = 5\), the values of \(d\) and \(\pi\) differ: \(d(5) = 5, \pi(5) = 20\). We will prove the curious fact that \(\pi(m)\) is always even, when \(m > 2\). In fact, we will give two proofs: one will reveal why the phenomenon occurs; the other will be the proof ‘from the book’: so delightfully concise and cunning that it gives nothing away!

**Proof 1** (assumes existence of \(d(m)\) and \(\pi(m)\))

Observe that, if we pair off entries over a complete period from \(f_0\) to \(f_{\pi(m)}\), counting inwards, the paired entries alternate to 0 (mod \(m\)) (evens), and are equal (odds). E.g. for \(m = 5\) we have:

\[
\begin{array}{cccccccc}
0 & 1 & 1 & 2 & 3 & 0 & 3 & 3 \\
\hline
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
\end{array}
\]

Now suppose \(\pi(m)\) is odd, say \(\pi(m) = 2k + 1\). Then counting into the middle of the first period we will finish with a middle pair: \(f_k, f_{k+1}\), as shown below.

\[
\begin{array}{cccccccc}
0 & 1 & \cdots & k & k+1 & k+2 & 2k+1 \\
\hline
\cdots & f_{k-1} & f_k & f_{k+1} & f_{k+2} & \cdots & 1 & 0
\end{array}
\]

Now if \(k\) is even then \(f_k + f_{k+1} = 0\) (mod \(m\)). Then \(f_{k+2} = 0\) since, by definition, \(f_{k+2} = f_k + f_{k+1}\) (mod \(m\)). But then \(f_{k-1} = 0\) because \(f_{k-1} + f_k\) form an odd pairing. And if \(k\) is odd then \(f_k = f_{k+1}\). But since \(f_{k+1} = f_k + f_{k-1}\) this means that \(f_{k-1} = 0\). Now \(f_{k+2} = 0\) because \(f_{k-1} + f_{k+1}\) are an even pairing and sum to 0 (mod \(m\)). Either way, we have found a subsequent ‘0 \times 0’ and we conclude that \(d(m) = 1\) or \(d(m) = 3\). This means that \(F_1 = 1 = 0\) (mod \(m\)) or \(F_2 = 2 = 0\) (mod \(m\)), respectively.

**Proof 2** (assumes existence of \(\pi(m)\))

Let \(Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\), the so-called ‘Fibonacci matrix’. It is easy to confirm that \(Q^k = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}\). Then 1 = det\(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) = det\(Q^{\pi(m)}\) (mod \(m\)) = det\(Q^{\pi(m)}\) (mod \(m\)) = \((-1)^{\pi(m)}\) (mod \(m\)), which implies that \(\pi(m)\) is even or \(m = 1\) or 2.

The periodicity of the ‘reduced’ Fibonacci sequence was known to Lagrange in the 1780s and its divisibility properties were studied by Édouard Lucas in the 1870s. The modular periodicity of linear recurrences in general was systematically studied by Robert Carl Mihael in the 1920s and Morgan Ward in the 1930s. This theorem, as stated, is due to Donald Dines Wall (1960) who also gave Proof 1 above. The book proof shown here (Proof 2) is due to David Singerman.

Web link: webspace.ship.edu/msrenault/fibonacci/fib.htm (contains another proof ‘from the book’ of even periodicity).