THEOREM OF THE DAY

Singmaster’s Binomial Multiplicity Bound (A Theorem under Construction!)  For integer $k > 1$, let $N(k)$ denote the multiplicity of $k$ as a binomial coefficient; i.e. $N(k) = \left\lfloor (n, r) \in \mathbb{Z}^2 : n = \binom{k}{r} \right\rfloor$. Then $N(k) = O(\log k)$.

Each row of Pascal’s Triangle appears here as a triangle; in fact the complete rows would be rhomboids since they are symmetric about their centres; the last row shown in its entirety on the right is row 6 (counting from 0):

- 1 6 15 20 15 6 1

Observing that, apart from 1, the $k$-th row contains no numbers smaller than $k$, it is apparent that every positive integer $k$ must have finite multiplicity. So the $N(k)$ of the theorem is a well-defined number; but David Singmaster has conjectured the ultimate: $N(k) = O(1)$, i.e. there is an absolute constant $C$ such that $N(k) \leq C$ for all integers $k > 1$. To date, $N(k) = 6$ is the largest value for which an infinite family of occurrences has been discovered: the Fibonacci numbers $F_n$, 1, 1, 2, 3, 5, 8, 13, … (which appear as sums of diagonals of Pascal’s Triangle) supply, for any even $n$:

$$F_n F_{n+1} - F_{n+2} F_{n-1} = F_n (F_{n+1} - F_{n-1}) = F_n F_{n+1} - F_{n-1} F_{n+1}.$$

Thus, for $n = 4$, $(3^3 \times 5 - 2 \times 3) = (3^3 \times 2^2 - 3^3) = 3003$. In addition to row 3003 it happens that 3003 has one further occurrence, giving $N(3003) = 8$ (four rows each with two occurrences), the largest known value.

CONSTRUCTION NOTES 1971: Singmaster proves $N(k) = O(\log k)$; checks $N(k) \leq 8$ up to $k = 2^{23}$ (later $k = 2^{48}$).
1974: Harvey Abbott, Paul Erdős and Denis Hanson prove $N(k) = O(\log^2 k / \log \log k)$.
2004: Daniel M. Kane proves $N(k) = O(\log k \log \log k / (\log \log k)^2)$.
2007: Kane increases his denoominator to $(\log \log k)^3$.

Proof of $O(\log k)$ bound:

$N(k) =$ no. of solutions of $k = \binom{a+b}{a} = \binom{a+b}{b}, 1 \leq a, b \leq k$ in $\mathbb{Z}$. Since $\binom{a+b}{a}$ increases in each of $a$ and $b$, any choice of $a$ or $b$, admits at most one new solution value for $N(k)$. Now suppose that $s$ satisfies $k < \binom{s}{2}$; then $k = \binom{a+b}{a}$ implies that $a$ or $b < s$, so the solution count $N(k) \leq 2 \times s$. Take the least such $s$. Then $2^{s+1} \leq 2(s+1) \leq k$, so the solution count $N(k) \leq 2 \log k$.

Web link: aperiodical.com/2013/01/open-season-singmasters-conjecture/

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