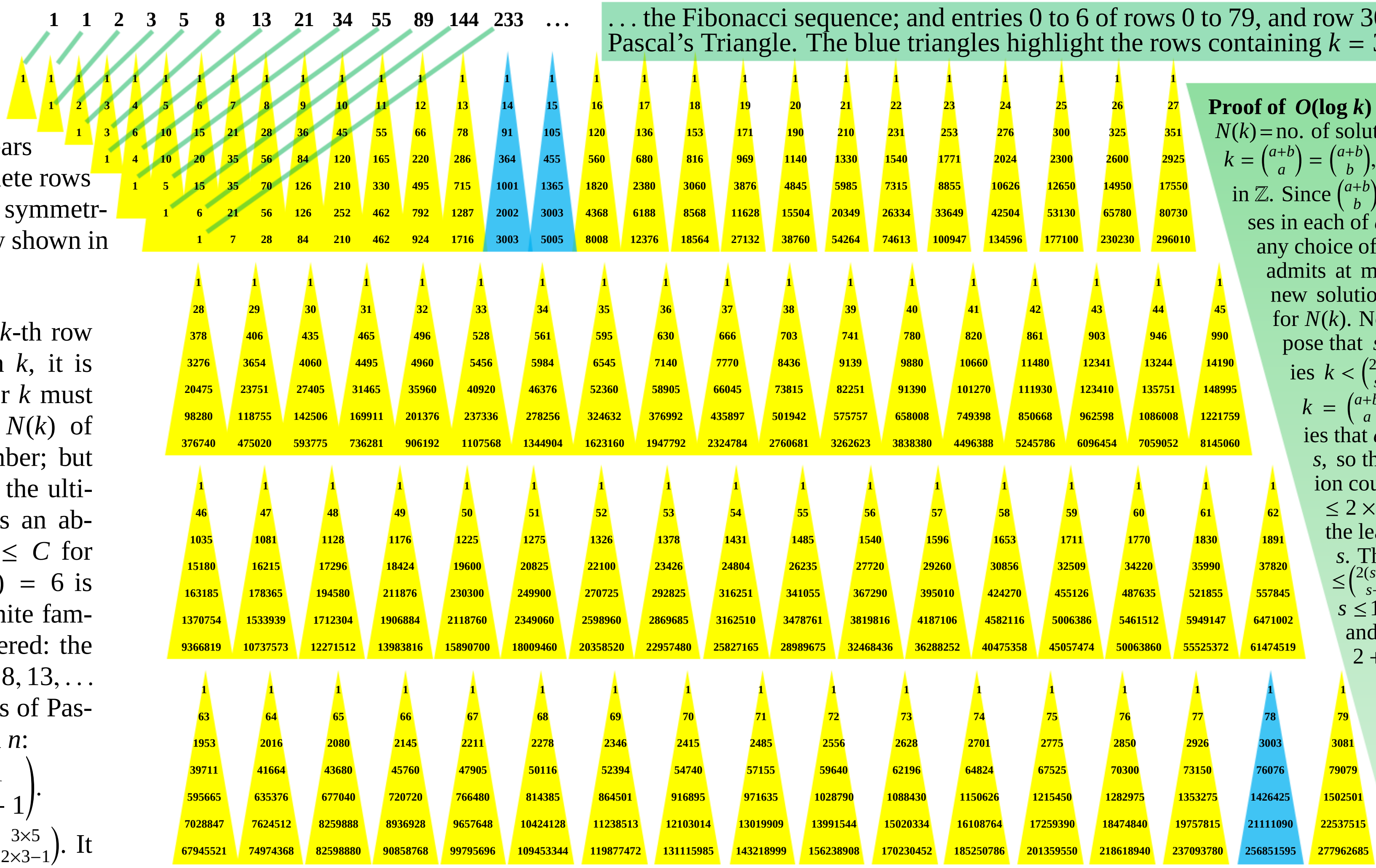
THEOREM OF THE DAY

Singmaster's Binomial Multiplicity Bound (A Theorem under Construction!) *For integer $k > 1$, let $N(k)$ denote the multiplicity of k as a binomial coefficient; i.e. $N(k) = \left|\left\{(n,r) \in \mathbb{Z}^2 : k = \binom{n}{r}\right\}\right|$. Then $N(k) = O(\log k)$.*



… the Fibonacci sequence; and entries 0 to 6 of rows 0 to 79, and row 3003, of Pascal's Triangle. The blue triangles highlight the rows containing $k = 3003$.

Each row of Pascal's Triangle appears here as a triangle; in fact the complete rows would be rhomboids since they are symmetrical about their centres; the last row shown in its entirety on the right is the sixth:

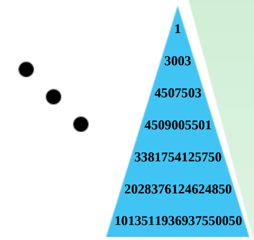
1 6 15 20 15 6 1.

Observing that, apart from 1, the k-th row contains no numbers smaller than k, it is apparent that every positive integer k must have finite multiplicity. So the $N(k)$ of the theorem is a well-defined number; but David Singmaster has conjectured the ultimate: $N(k) = O(1)$, i.e. there is an absolute constant C such that $N(k) \leq C$ for all integers $k > 1$. To date, $N(k) = 6$ is the largest value for which an infinite family of occurrences has been discovered: the Fibonacci numbers F_n, $1, 1, 2, 3, 5, 8, 13, \ldots$ (which appear as sums of diagonals of Pascal's Triangle) supply, for any even n:

$$N = \binom{F_nF_{n+1} - 1}{F_{n-1}F_n} = \binom{F_nF_{n+1}}{F_{n-1}F_n - 1}.$$

Thus, for $n = 4$, $3003 = \binom{3\times5-1}{2\times3} = \binom{3\times5}{2\times3-1}$. It happens that 3003 has one further occurrence, giving $N(3003) = 8$ (four rows each with two occurrences), the largest known value.

Proof of $O(\log k)$ bound: $N(k)$ = no. of solutions of $k = \binom{a+b}{a} = \binom{a+b}{b}$, $1 \leq a, b$ in $\mathbb{Z}$. Since $\binom{a+b}{b}$ increases in each of a and b, any choice of a, or b, admits at most one new solution value for $N(k)$. Now suppose that s satisfies $k < \binom{2s}{s}$; then $k = \binom{a+b}{a}$ implies that a or $b < s$, so the solution count $N(k) \leq 2 \times s$. Take the least such s. Then $2^{s-1} \leq \binom{2(s-1)}{s-1} \leq k$, $s \leq 1 + \log k$, and $N(k) \leq 2 + 2\log k = O(k)$. □



CONSTRUCTION NOTES 1971: Singmaster proves $N(k) = O(\log k)$; checks $N(k) \leq 8$ up to $k = 2^{23}$ (later $k = 2^{48}$).
1974: Harvey Abbott, Paul Erdös and Denis Hanson prove $N(k) = O(\log k / \log\log k)$.
2004: Daniel M. Kane proves $N(k) = O\left((\log k)\log\log\log k/(\log\log k)^2\right)$.
2007: Kane increases his denominator to $(\log\log k)^3$.

Web link: aperiodical.com/2013/01/open-season-singmasters-conjecture/

Further reading: Pascal's Arithmetical Triangle by A.W.F. Edwards, Johns Hopkins University Press, 2002.