



THEOREM OF THE DAY

Singmaster's Binomial Multiplicity Bound (A Theorem under Construction!) For integer $k > 1$, let $N(k)$ denote the multiplicity of k as a binomial coefficient; i.e. $N(k) = \left| \left\{ (n, r) \in \mathbb{Z}^2 : k = \binom{n}{r} \right\} \right|$. Then $N(k) = O(\log k)$.

... the Fibonacci sequence; and entries 0 to 6 of rows 0 to 79, and row 3003, of Pascal's Triangle. The blue triangles highlight the rows containing $k = 3003$.

Each row of Pascal's Triangle appears here as a triangle; in fact the complete rows would be rhomboids since they are symmetrical about their centres; the last row shown in its entirety on the right is row 6 (counting from 0):

$$1 \ 6 \ 15 \ 20 \ 15 \ 6 \ 1.$$

Observing that, apart from 1, the k -th row contains no numbers smaller than k , it is apparent that every positive integer k must have finite multiplicity. So the $N(k)$ of the theorem is a well-defined number; but David Singmaster has conjectured the ultimate: $N(k) = O(1)$, i.e. there is an absolute constant C such that $N(k) \leq C$ for all integers $k > 1$. To date, $N(k) = 6$ is the largest value for which an infinite family of occurrences has been discovered: the Fibonacci numbers F_n , 1, 1, 2, 3, 5, 8, 13, ... (which appear as sums of diagonals of Pascal's Triangle) supply, for any even n :

$$\binom{F_n F_{n+1} - 1}{F_{n-1} F_n} = \binom{F_n F_{n+1}}{F_{n-1} F_n - 1}.$$

Thus, for $n = 4$, $\binom{3 \times 5 - 1}{2 \times 3} = \binom{3 \times 5}{2 \times 3 - 1} = 3003$. In

addition to row 3003 it happens that 3003 has one further occurrence, giving $N(3003) = 8$ (four rows each with two occurrences), the largest known value.



CONSTRUCTION NOTES 1971: Singmaster proves $N(k) = O(\log k)$; checks $N(k) \leq 8$ up to $k = 2^{23}$ (later $k = 2^{48}$).

1974: Harvey Abbott, Paul Erdős and Denis Hanson prove $N(k) = O(\log k / \log \log k)$.

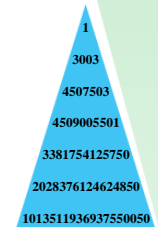
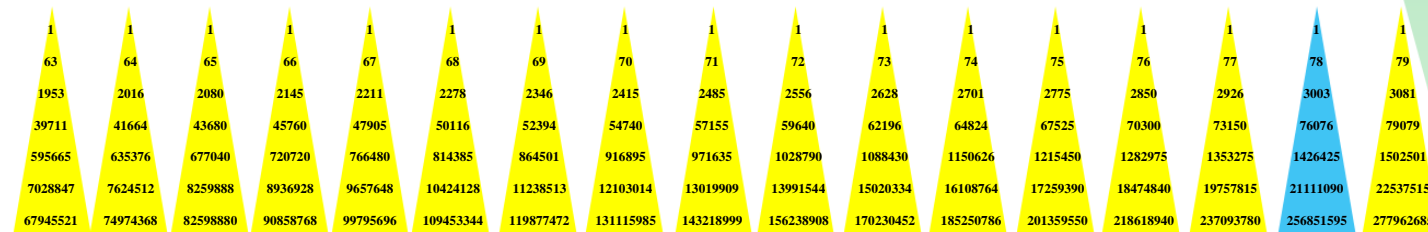
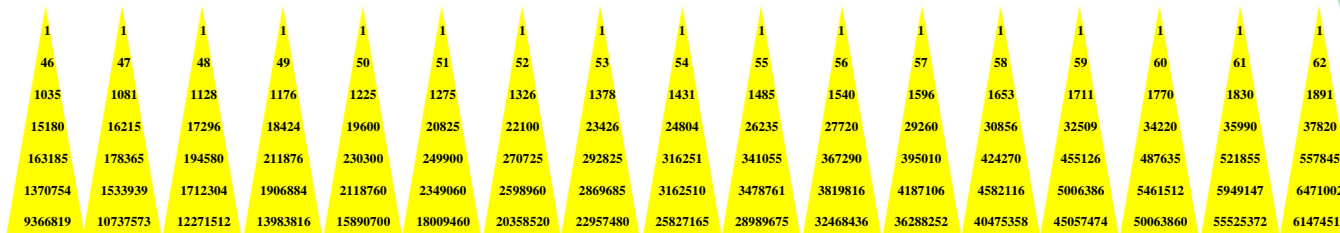
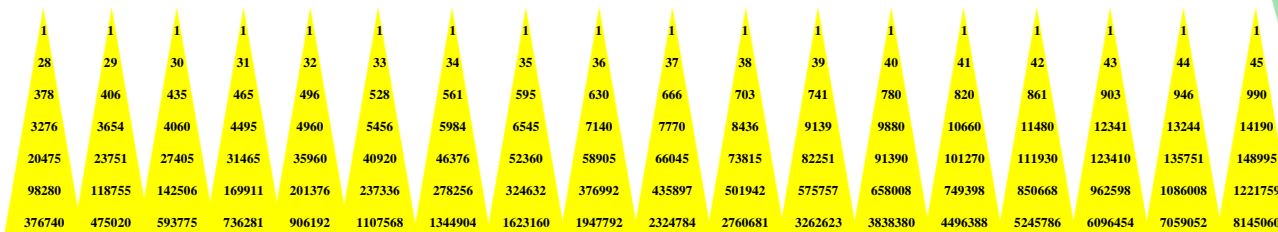
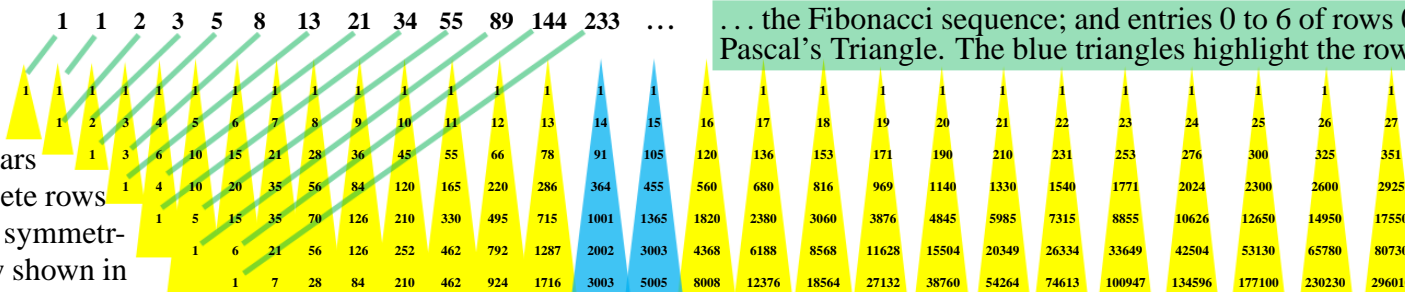
2004: Daniel M. Kane proves $N(k) = O((\log k) \log \log \log k / (\log \log k)^2)$.

2007: Kane increases his denominator to $(\log \log k)^3$.

Further updates via notes button



Proof of $O(\log k)$ bound:
 $N(k)$ = no. of solutions of $k = \binom{a+b}{a} = \binom{a+b}{b}$, $1 \leq a, b$ in \mathbb{Z} . Since $\binom{a+b}{b}$ increases in each of a and b , any choice of a , or b , admits at most one new solution value for $N(k)$. Now suppose that s satisfies $k < \binom{2s}{s}$; then $k = \binom{a+b}{a}$ implies that a or $b < s$, so the solution count $N(k) \leq 2 \times s$. Take the least such s . Then $2^{s-1} \leq \binom{2(s-1)}{s-1} \leq k$, $s \leq 1 + \log k$, and $N(k) \leq 2 + 2 \log k = O(k)$. \square



Web link: aperiodical.com/2013/01/open-season-singmasters-conjecture/

Further reading: *Pascal's Arithmetical Triangle* by A.W.F. Edwards, Johns Hopkins University Press, 2002.