



# THEOREM OF THE DAY



**Stirling's Approximation** For positive integers  $n$ , the value of the factorial function  $n!$  is given asymptotically by

$$n! \approx \sqrt{\tau n} n^n e^{-n},$$

where  $\tau = 2\pi$ .



Pascal's triangle $n = 0 \dots 12$ .	Using Stirling's approximation (& rounding)	Ratio of actual to unrounded approximation
1	1	1
1, 1	1, 1	1, 1
1, 2, 1	1, 2, 1	1, 0.885, 1
1, 3, 3, 1	1, 3, 3, 1	1, 0.909, 0.909
1, 4, 6, 4, 1	1, 4, 6, 4, 1	1, 0.917, 0.940, 0.917
1, 5, 10, 10, 5, 1	1, 5, 11, 11, 5, 1	1, 0.916, 0.943, 0.943
1, 5, 20, 15, 6, 1	1, 7, 16, 21, 16, 7, 1	1, 0.916, 0.955, 0.962, 0.955
1, 35, 35, 21, 7, 1	1, 8, 22, 36, 36, 22, 8, 1	1, 0.920, 0.955, 0.964, 0.964
1, 56, 70, 56, 28, 8, 1	1, 9, 29, 58, 72, 58, 29, 9, 1	1, 0.916, 0.956, 0.966, 0.968, 0.956
1, 126, 126, 84, 36, 9, 1	1, 10, 38, 87, 130, 130, 87, 38, 10, 1	1, 0.921, 0.952, 0.966, 0.969, 0.969
1, 10, 252, 210, 120, 45, 10, 1	1, 11, 47, 124, 216, 258, 216, 124, 47, 11, 1	1, 0.917, 0.949, 0.976, 0.972, 0.969, 0.976
1, 462, 462, 330, 165, 55, 11, 1	1, 12, 57, 170, 338, 473, 473, 338, 170, 57, 12, 1	1, 0.924, 0.953, 0.971, 0.973, 0.981, 0.981
1, 92, 924, 792, 495, 220, 66, 12, 1	1, 13, 69, 227, 507, 809, 943, 809, 507, 227, 69, 13, 1	1, 0.923, 0.955, 0.973, 0.976, 0.977, 0.980, 0.977

Pascal's triangle  $n = 0 \dots 12$ .

Using Stirling's approximation (& rounding)

Ratio of actual to unrounded approximation

Since binomial coefficients  $\binom{n}{k}$  can be calculated as  $\binom{n}{k} = n!/k!(n-k)!$  we can use Stirling's approximation to calculate the approximate values in Pascal's triangle, as shown here. The boxed values locate the so-called **central binomial coefficients**, for which our approximation simplifies neatly to  $\binom{n}{n/2} \sim 2^{n+1}/\sqrt{\tau n}$ . They indicate that Stirling's approximation is increasing in accuracy as  $n$  increases. The distance between the actual and approximated values grows larger with  $n$  but their ratio grows closer and closer to one so that, *asymptotically*, the approximation is accurate. The approximation may also be stated as an equality  $n! = \sqrt{\tau n}(n/e)^n(1 + O(1/n))$ . The  $O(1/n)$  represents additional terms which vanish as quickly as or quicker than  $1/n$ ; it may be 'expanded' as far as desired: Knuth replaces  $1 + O(1/n)$  with  $1 + 1/(12n) + 1/(288n^2) - 139/(51840n^3) - 571/(2488320n^4) + O(1/n^5)$  which, rounding to the nearest integer, gives the exact value of  $n!$  up to  $n = 12$ .

James Stirling published his approximation in 1730. It would seem more fair to call it the De Moivre–Stirling approximation since Abraham de Moivre had just discovered that  $n! \approx c \sqrt{n} n^n e^{-n}$ , for some constant  $c$ . Stirling's contribution, acknowledged by De Moivre, was to identify the constant as  $\sqrt{\tau}$ . The formula thereby becomes one of those rare mathematical beasts (like Euler's  $e^{\tau i/2} + 1 = 0$ ) that display in a natural way the fundamental constants  $e$  and  $\tau$ .

**Web link:** [www.stat.ualberta.ca/people/schmu/preprints/factorial.pdf](http://www.stat.ualberta.ca/people/schmu/preprints/factorial.pdf)

**Further reading:** *The Art of Computer Programming* by Donald E. Knuth, Addison-Wesley, 1999 edition. Vol. 1, section 1.2.11.

