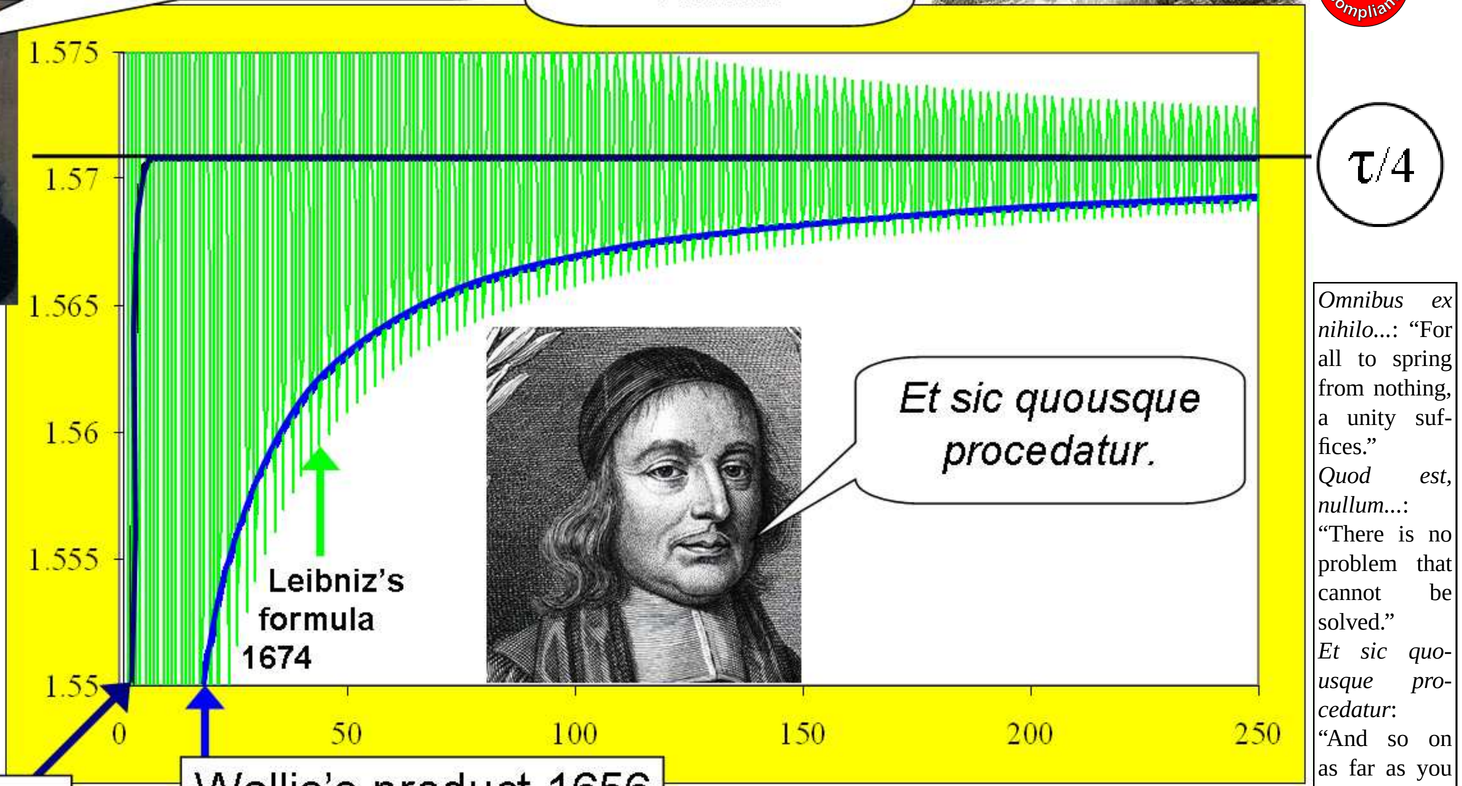
THEOREM OF THE DAY

Wallis's Product *The value of $\tau/4$ $(\tau = 2\pi)$ is given by the infinite product*

$$\prod_{k=1}^{\infty} \frac{(2k)^2}{(2k-1)(2k+1)} = \frac{2^2}{1.3} \cdot \frac{4^2}{3.5} \cdot \frac{6^2}{5.7} \cdots .$$

The graph on the right contrasts three different derivations of $\tau/4 \approx 1.5708$. Ironically, the older the method of approximation the more swiftly it appears to converge! Viète's formula, though, involves taking square roots, approximating one irrational number, τ, by others. The infinite product of John Wallis (1606–1703) achieves the same goal using only rational numbers. The Leibniz formula, meanwhile, is a summation, not a product: $\tau/4 = 2 - 2/3 + 2/5 - 2/7 + \ldots$ The alternating signs account for the oscillation of its curve above and below the value of $\tau/4$. And as a matter of fact, this summation was known to the Indian scholar Madhava of Sangamagrama in the fifteenth century. Wallis's product arose out of his attempts to solve the notorious problem of squaring the circle; with hindsight and with the calculus (which Wallis helped to originate) it is derived effortlessly by putting $x = \tau/4$ in an infinite product expansion for $\sin x$ due to Leonard Euler: $\sin x = x \prod_{n=1}^{\infty} (1 - (2x/n\tau)^2)$.



Omnibus ex nihilo...: "For all to spring from nothing, a unity suffices." *Quod est, nullum...*: "There is no problem that cannot be solved." *Et sic quousque procedatur*: "And so on as far as you like"

Wallis derived his product through a 'method of interpolation', Proposition 191 in his 1656 *Arithmetica Infinitorum*, described by the scholar Jacqueline A. Stedall as "perhaps the one real stroke of genius in Wallis's long mathematical career."

Web link: www.maa.org/press/periodicals/convergence/index-to-mathematical-treasures (the first Wallis entry). The images, from left to right, of Leibniz, Wallis and Viète are from the biographical articles at en.wikipedia.org/.

Further reading: *The Arithmetic of Infinitesimals: John Wallis 1656* by Jacqueline A. Stedall, Springer, 2004.