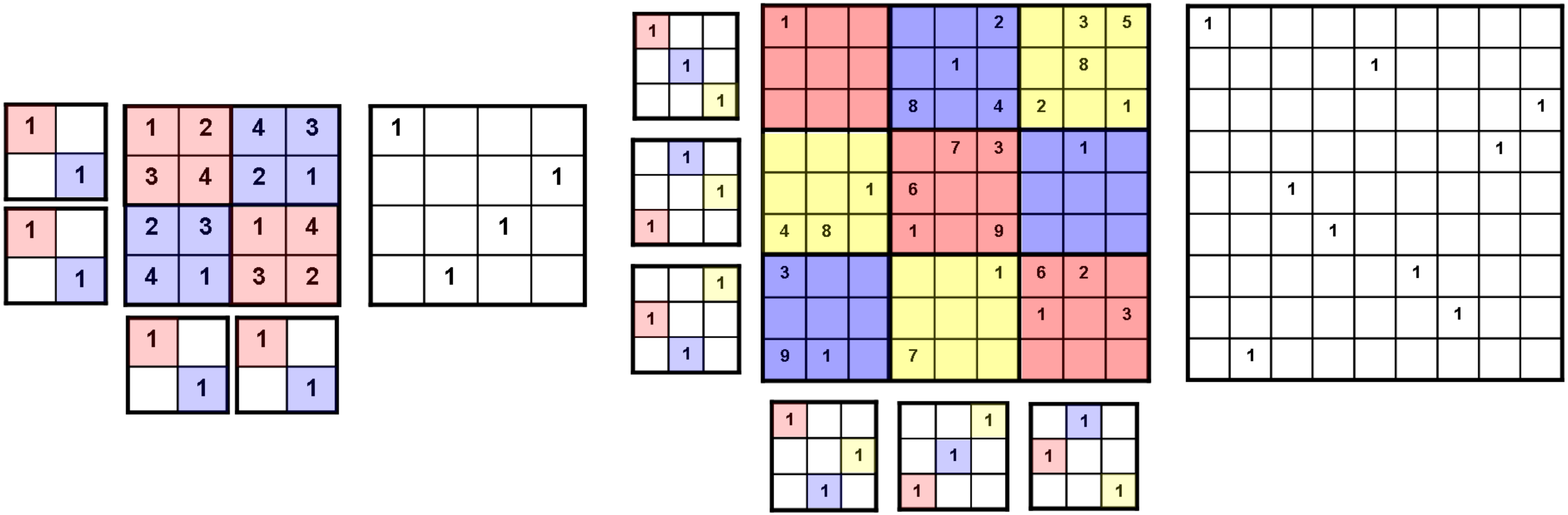
# THEOREM OF THE DAY

**The Birkhoff–von Neumann Theorem** *The set of $n \times n$ doubly stochastic matrices forms a convex polytope whose vertices are the $n \times n$ permutation matrices.*



A *doubly stochastic* matrix is one in which every row and column consists of non-negative reals adding to one. Permutation matrices are a very special case: they can be thought of as $n \times n$ grids, empty except for a one appearing exactly once in each row and column. Sudoku puzzles are a good source of $9 \times 9$ permutation matrices, since the ones in any completed puzzle satisfy this defining property. The puzzles shown above, $4 \times 4$ as well as $9 \times 9$, illustrate this twice over: they have been carefully constructed so that each set of horizontal and vertical blocks produces a smaller permutation matrix. Complete the $9 \times 9$ square (by applying your sudoku skills or by going to a website such as www.dkmsoftware.com/sudoku): you will find it is even more special in this respect.

A *convex polytope* is the region of $n$-dimensional space that is 'enclosed' by a given collection of points — some or all of these will be the extreme points or *vertices* of the polytope, no part of the region lying 'beyond' them. This is made precise by saying that any point in the polytope may be written as a weighted sum of the vertices, the weights lying in the interval $[0, 1]$.

Garrett Birkhoff's 1946 theorem, discovered independently by John von Neumann in 1953, is important in many ways, not least because it means that ordering a set of objects, in order to minimise a linear function defined on them, can be solved rapidly as a linear programming problem.

**Web link:** www-igm.univ-mlv.fr/~fpsac/FPSAC07/SITE07/conpro.htm: click on Jessica Striker's paper.

**Further reading:** *Combinatorial Optimization: Polyhedra and Efficiency* by Alexander Schrijver, Springer, Berlin, 2002.