The Birkhoff–von Neumann Theorem: The set of $n \times n$ doubly stochastic matrices forms a convex polytope whose vertices are the $n \times n$ permutation matrices.

A doubly stochastic matrix is one in which every row and column consists of non-negative reals adding to one. Permutation matrices are a very special case: they can be thought of as $n \times n$ grids, empty except for a one appearing exactly once in each row and column. Sudoku puzzles are a good source of $9 \times 9$ permutation matrices, since the ones in any completed puzzle satisfy this defining property. The puzzles shown above, $4 \times 4$ as well as $9 \times 9$, illustrate this twice over: they have been carefully constructed so that each set of horizontal and vertical blocks produces a smaller permutation matrix. Complete the $9 \times 9$ square (by applying your sudoku skills or by going to a website such as www.dkmsoftware.com/sudoku): you will find it is even more special in this respect.

A convex polytope is the region of $n$-dimensional space that is ‘enclosed’ by a given collection of points — some or all of these will be the extreme points or vertices of the polytope, no part of the region lying ‘beyond’ them. This is made precise by saying that any point in the polytope may be written as a weighted sum of the vertices, the weights lying in the interval $[0, 1]$.

Garrett Birkhoff’s 1946 theorem, discovered independently by John von Neumann in 1953, is important in many ways, not least because it means that ordering a set of objects, in order to minimise a linear function defined on them, can be solved rapidly as a linear programming problem.

Web link: www-igm.univ-mlv.fr/~fpsacFPSAC07/SITE07/conpro.htm: click on Jessica Striker’s paper.