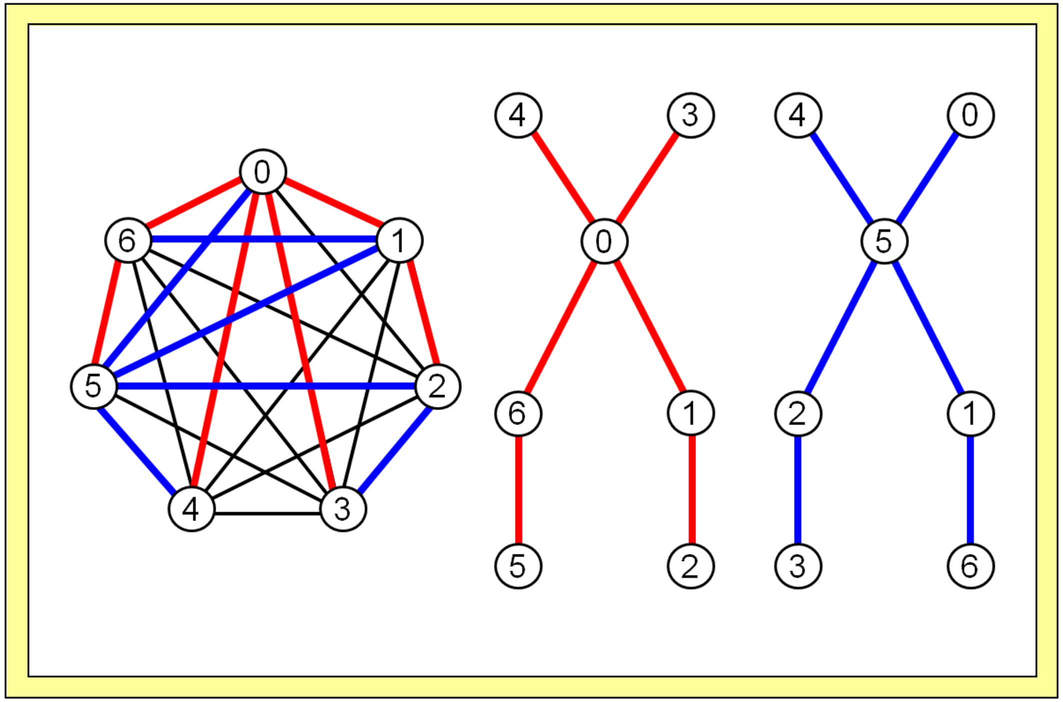
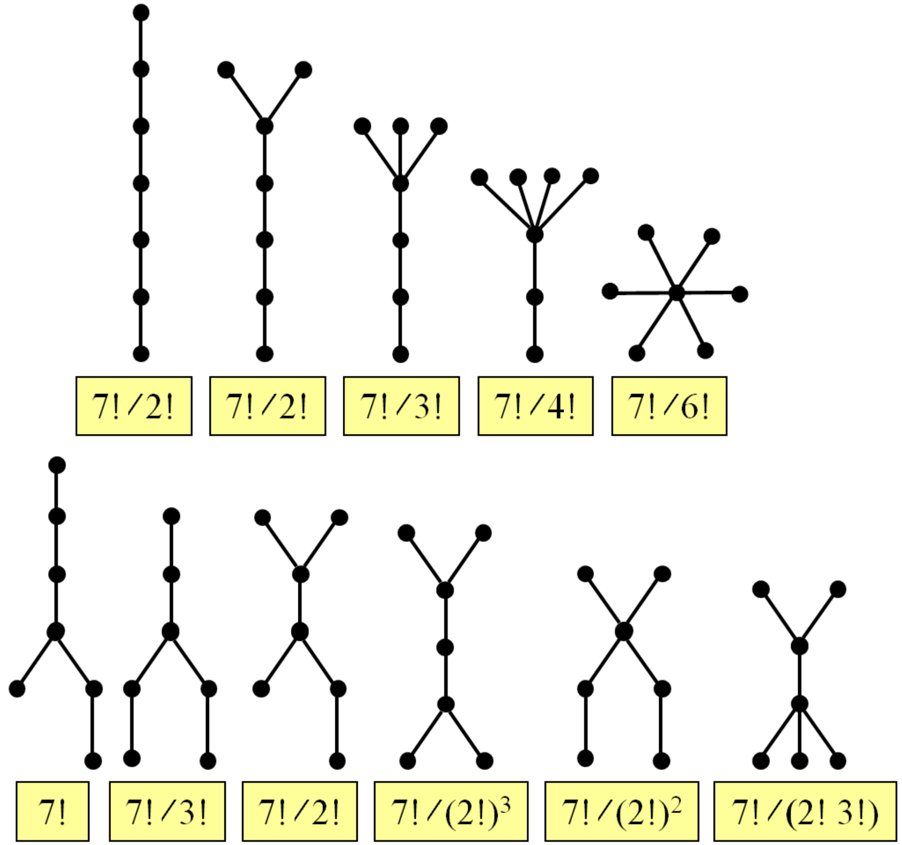
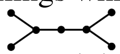
THEOREM OF THE DAY

Cayley's Formula *The number of labelled trees on n vertices, $n \geq 1$, is n^{n-2}.*





Labelled trees on n vertices may be thought of as those subgraphs of the labelled complete graph K_n which have $n-1$ edges and no cycles. The two trees shown on the left are distinct subgraphs of K_7 although they are isomorphic as *unlabelled* trees. There are eleven non-isomorphic unlabelled trees and these are shown on the right together with the number of distinct labellings which each admits. These numbers are calculated as 7! (that is, all possible labellings) divided by the number of symmetries in the tree. In , for example, either of the end pairs of vertices may be permuted internally or the complete pairs may be reflected about the central vertex, giving $2 \times 2 \times 2$ possible distinct permutations in total. There are thus $7!/(2!)^3 = 630$ distinct labellings of this tree. The possible labellings over all eleven unlabelled trees gives a grand total of $7^5 = 16807$.

The English mathematician Arthur Cayley (1821–1895) published this formula in 1889. At the time he was working on permutation groups and on invariant theory and its relationship to symmetric functions. His famous formula may have arisen out of these studies, but R.P. Stanley has noted that it was already known to Sylvester and Borchardt.

Web link: www-math.mit.edu/~rstan/algcomb/index.html. See Chapter 9 (the appendix has some nice combinatorial proofs and historical notes).
Further reading: *Proofs from the Book,* by Martin Aigner and Günter M. Ziegler, Springer-Verlag, Berlin, 5th Edition, 2014, chapter 32.