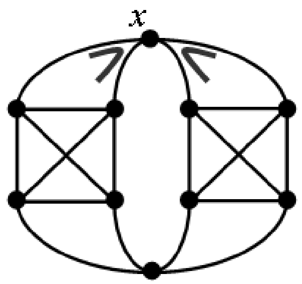
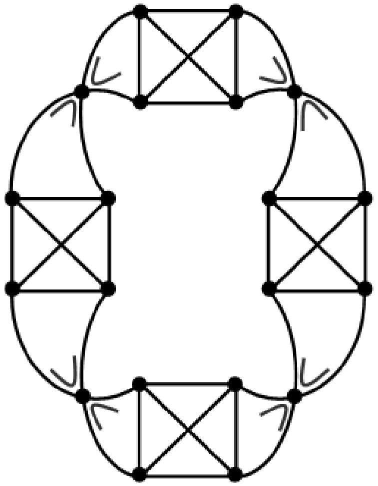
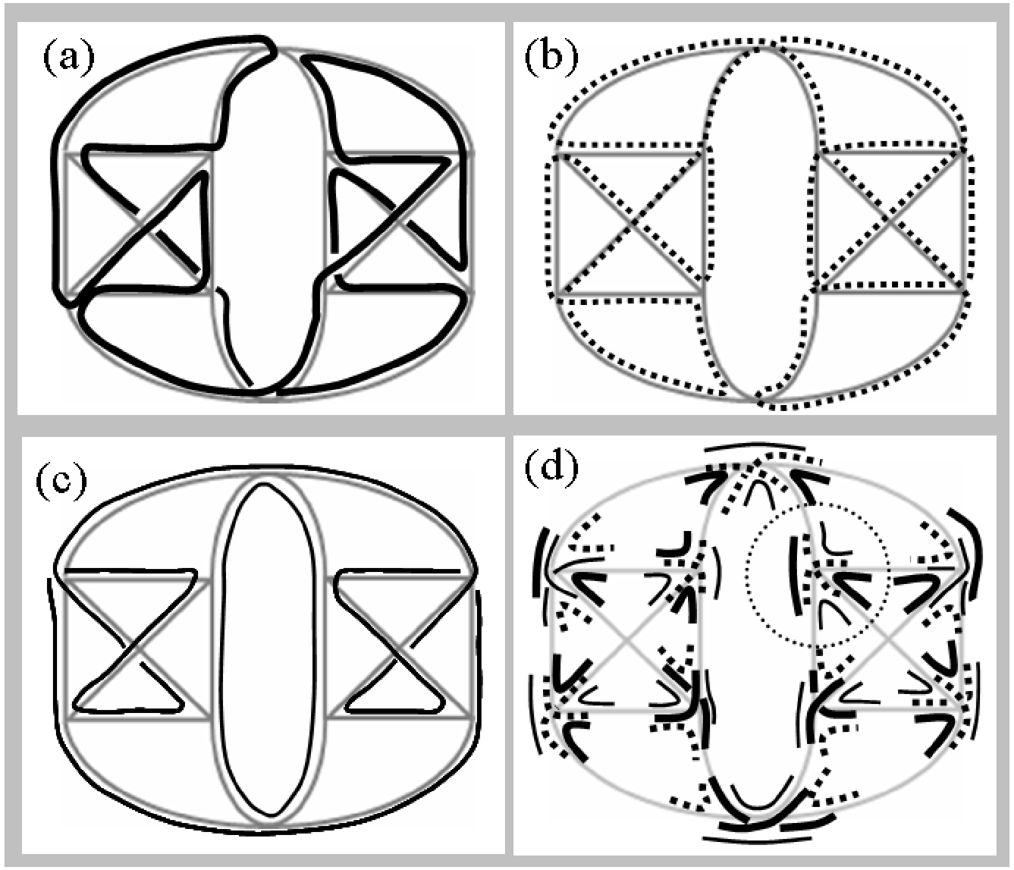
THEOREM OF THE DAY

Jackson's Theorem on Compatible Euler Tours *A connected 4-regular graph G admits three pairwise compatible Euler tours if and only if, for any set B of bitransitions, the number of components ω_B induced by splitting each vertex on B satisfies* $3(\omega_B - 1) \leq 2|B|$.

The graphs shown here are *Eulerian*, that is, admitting an *Euler tour*: a sequence of consecutive edges passing every edge of the graph exactly once and finishing at its start vertex. They are also *4-regular*, all vertices being incident with exactly four edges; in such graphs any Euler tour must pass each vertex exactly twice, via two disjoint consecutive edge pairs. These pairs are said to form a *bitransition*. In the top-left graph, for instance, at vertex x a bitransition is shown which arrives and leaves by the left-hand edges on one visit, and is thus obliged to arrive and leave by the right-hand edges on the other visit. A complete Euler tour including this bitransition is shown at (a). At (b) every bitransition is *compatible* with (a), meaning that at every vertex a bipartition different from (a) is chosen. And now at each vertex there remains only one possible bitransition compatible with both (a) and (b); this is shown at (c). Unfortunately we have not produced three pairwise compatible Euler tours: the bitransitions at (c) split the graph into two separate cycles, an outer one on 14 edges and an inner one on 6. We say that *splitting* the six vertices on the bipartitions around this inner cycle induces two components. This is repaired at (d) where the circled vertex has exchanged bipartitions between (a) and (c). The bipartitions at (d) specify three compatible Euler tours. Is this always possible?







Jackson's theorem says no: in the bottom-left graph a set B of 4 bitransitions has been indicated which separates the graph into $\omega_B = 4$ components, in the same manner as the bitransitions in (c) induced a separation into two. Now $3(\omega_B - 1) = 3 \times (4 - 1) = 9 > 2 \times 4 = 2|B|$, so three pairwise compatible Euler tours are impossible in this graph.

This is a classic deployment of Jack Edmonds' idea of a *good characterisation*: we can certify the existence of some object just by exhibiting it; to certify *nonexistence* we need to exhibit something else which *precludes* existence. Bill Jackson's 1991 theorem provides such a non-existence certificate for three compatible Euler tours: a suitable set of bitransitions.

Web link: users.utu.fi/harju/graphtheory/graphtheory.pdf

Further reading: *Graph Theory* by J.A. Bondy and U.S.R. Murty, Springer, 2008.